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Boundary-value problems in plane strain non-linear elastostatics.

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BOUNDARY-VALUE PROBLEMS IN PLANE STRAIN NON-LINEAR ELASTOSTATICS

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Dedicated to:- Maxine who worried about everything

ABSTRACT:- The subject matter of this thesis lies within the confines of the mathematical study of finite deformation solid mechanics.

The pertinent equations of the field of study are first surveyed. Following this an ad-hoc solution method is developed and used to solve a variety of plane strain problems. The method is compared and contrasted with that of JOHN. A complex variable formulation is developed for finite deformation plane strain problems. Out of this formulation a general solution method for a significant class of materials results. This method is developed and applied to a variety of problems. For one problem the solution is developed to the stage where the deformation field may be plotted. The solution method is compared with that of **MUSKHELISHVILI**.

At all points in the thesis material limitations in the form of inequalities are highlighted where appropriate.

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CONTENTS

CHAPTER 0: INTRODUCTION

Within the confines of the small deformation (linear) theory of elasticity a great number of problems have been solved. This is not, however, the case when large deformations are admitted; as in the theory of finite or non-linear elasticity. It is within this latter context that the subject matter of this thesis lies. In particular, problems which admit a two dimensional description are considered, that is problems concerning non-linear material undergoing finite plane strain deformations.

Much work in finite elasticity adopts the simplifying assumption of incompressibility. Solutions to a number of so restricted problems are known. Conversely, when this assumption is not adopted and the elastic material is allowed to be compressible very few solutions are documented. The intention of this thesis is to explore possibilities for the solution of boundary-value problems for compressible isotropic materials within the context of plane strain.

The purpose of this thesis is to investigate a technique for solving two dimensional large deformation elastostatic problems. The technique developed is analytic in nature as are the solutions obtained. Clearly, numerical solutions are obtainable as is evidenced by the book by ODEN (1972). Such solutions are of value but the objective of an analytic solution is different. The aim here is to gain some insight into the fundamentals of material behaviour particularly when finite deformations are admitted. One such fundamental question is what constitutive laws may be admitted such that the corresponding material behaviour may be said to be physically reasonable. This particular question is addressed throughout this thesis. The numerical and analytical approaches are complementary.

In Chapter 1, a summary of the pertinent equations and concepts of finite elasticity theory are given. The notation is also introduced. Chapter 2 introduces a solution method to be developed. Initially the technique derived much motivation from that employed by F. JOHN (1960) for his defined class of harmonic materials. In Chapter 3 certain problems are solved, using the technique developed in Chapter 2. The problems are not of any paramount importance but the solutions are of theoretical interest. Their inclusion is purely to demonstrate and highlight characteristics of the solution method. The analytical solutions are employed to characterise inherent properties of the material class chosen. A version of the work presented in Chapters 2 and 3 has appeared in the International Journal of Solids and Strüctures 1977.

In Chapter 4 a complex variable formulation is employed. All the equations of Chapter 2 are posed in terms of a Lagrangian complex coordinate z and its conjugate $\bar{z}$. The equations are seen to take a particularly simple form for materials of the harmonic type and a general solution is obtainable. The solution generated is in terms of two arbitrary functions analytic in z and $\bar{z}$ respectively. These arbitrary functions are shown to be determined for some specific boundary-value problems. Solutions obtained are discussed with particular emphasis being placed on volume change characteristics. Finally, in this chapter a small strain asymptotic analysis is employed and the solution technique is shown to reduce to that of **MUSKHELISHVILI** (1963).

In Chapter 5, the problem of an infinite plane with a rigid circular inclusion subjected to a uniaxial tension applied at infinity is considered in some detail. Other boundary-value problems concerning a circular anomaly in an infinite plate are also considered as are problems pertaining to annuli. Throughout this and Chapter 4, constitutive restrictions are highlighted when appropriate.

In Chapter 6, the solution of the problem specified above and which is solved in detail in Chapter 5 is again considered. A closed form analytic solution is obtained for the "semi-linear" material class. The solution is continued numerically. In particular. the deformation field, as represented by the deformation of an imbedded mesh, is presented diagrammatically for various applied traction values. Singularities and branch points of the solution field are discussed in some considerable detail.

The work surveyed in Chapters 4 and 5 has been published in the RHEOLOGICA ACTA VOL.16, NO.2.

In order to place the present work in true perspective, the relevant work of previous authors is discussed at all stages. Where necessary their results are included in sufficient detail to afford comparison and preserve continuity. Names of authors in capitals, followed by a date, indicate the appropriate references which are listed alphabetically at the end of this thesis. Every effort has been made to make this dissertation self-contained, but where it is felt greater depth might be useful, a pertinent reference is given.

CHAPTER 1: GENERAL ELASTICITY THEORY

SECTION 1.1 INTRODUCTION

In this chapter the concepts and equations underlying the study of solid mechanics are presented. There are many texts wherein may be found a full and comprehensive development of the theory; TRUESDELL & NOLL (1965), GREEN & ZERNA (1954), PRAGER (1973), JAUNZEMIS (1967) and CHADWICK (1976) are but a selection. Texts of this type are very general in scope, whereas in this thesis a restricted field of interest, that of elastostatics, will be considered and, as such, the theory presented will be tailored accordingly

The presentation of the theory of elastostatics given in this chapter will be of a summary nature, serving to introduce the notation to be employed and to bring to the fore the underlying concepts. Where clarification is required, a reference to one of the above texts will normally be given, while in other cases the theory will be expanded in more detail.

In this thesis the materials considered are taken to be perfectly elastic, and they are presumed to be both isotropic and homogeneous. All processes considered are taken to be quasi-static and isothermal. All spatial co-ordinates are referred to a single rectangular Cartesian background frame.

SECTION 1.2 KINEMATICS

The co-ordinates X_μ and x_i respectively, denote the rectangular Cartesian co-ordinates of a typical material point in the reference and in the deformed configuration of the material. The co-ordinates are conventionally referred to as the Lagrangian and Eulerian variables respectively, and in this thesis they are referred to the same background frame. The reference configuration is taken to be both unstressed and strain free. The map

$$x_i \equiv x_i\ (X_\mu), \qquad 1.2.1$$

serves to define the deformation, where X_μ may range over the region occupied by the material.

There are many deformation measures that may be defined as evidenced by HILL [1968(a)(b)], who discussed in detail a particular class of such measures. One of the simplest of these measures,

$$\alpha_{i\mu} \equiv \partial x_i / \partial X_\mu \equiv x_{i,\mu} \quad , \qquad 1.2.2$$

is the deformation gradient and is adopted in this thesis. This measure satisfies many basic requirements, the most rudimentary being that it is a measure of local changes in length. It also satisfies the requirement of Galilean invariance (JAUNZEMIS [1967, p.191]). Other commonly employed deformation measures are as introduced in HILL [1968(a)],

$$\begin{aligned} \underline{e}^{(m)} &= \frac{1}{2m} \left((\underline{\alpha}^T \underline{\alpha})^m - \underline{I} \right) \quad \forall\, m \neq 0 \\ e^{(0)} &= \ln (\underline{\alpha}^T \underline{\alpha}) \quad . \end{aligned} \qquad 1.2.3$$

These reduce to that employed by the classical theories, for small deformations.

As a consequence of the characteristics of the field of interest examined here, the mapping 1.2.1 is required to be topological in nature. A sufficient condition for this is that the deformation gradient should exist almost everywhere and that the Jacobian J, should be non-zero. However,

$$J = \det (\alpha_{i\mu}) > 0 \qquad 1.2.4$$

is adopted, as this ensures that the deformed to undeformed volume ratio is positive.

The so-called Polar Decomposition Theorem (see CHADWICK [1976, p.33]) may be employed to decompose the $\alpha_{i\mu}$ of 1.2.1, as

$$\begin{aligned} [\alpha_{i\mu}] &= \underline{R}\,\underline{U} \qquad (= r_{ip} u_{p\mu}) \\ &= \underline{V}\,\underline{R} \qquad (= v_{ip} r_{p\mu}) \quad , \end{aligned} \qquad 1.2.5$$

where $\underline{R}$ is a rotation. This decomposition, 1.2.5, serves to define $\underline{U}$ and $\underline{V}$ which are both positive definite and symmetric, representing pure stretches. $\underline{U}$ and $\underline{V}$ are commonly called the right and left Green deformation measures respectively. A fuller discussion of these may be found in JAUNZEMIS (1967, p.150).

SECTION 1.3 STRESS

Denoting by t_i the components of the traction vector at a point on a particular surface with unit normal Nj, then

$$t_i = \sigma_{ij} N_j \qquad 1.3.1$$

is a statement of Cauchy's Fundamental Theorem (see JAUNZEMIS [1967, Sect.10]). This serves to introduce the Cauchy stress tensor σ_{ij}, which may be shown to be symmetric. The interpretation of each element σ_{ij} is that of the force per unit area in the i-th direction on a surface with a normal in the j-th. The Cauchy stress measure $[\sigma_{ij}]$, is conventionally employed, because it has a simple interpretation. The relationship 1.3.1 is in accord with the Principle of Local State (TRUESDELL & NOLL (1965), Section 267).

The Cauchy stress measure is by no means the only one, in fact it is possible to define an infinite number. In section 1.4 it will be shown how a conjugate stress measure may be defined for each of the strain measures of 1.2.3. In this dissertation two further stress measures are employed:-

(i) The Nominal or First Piola-Kirchoff stress measure $\underline{S}$ with components $S_{\mu i}$: each $S_{\mu i}$ represents the i-th component of traction on a unit undeformed area with an undeformed normal in the μ-th direction. The use of this measure eases the application of a Lagrangian description and it may be shown to be conjugate to the deformation gradient of equation 1.2.2.

(ii) The Symmetrized Biot Stress measure $t_{\mu\nu}$. This measure has no simple interpretation but it is conjugate to $U_{\mu\nu}$, the right stretch tensor of equation 1.2.5.

The first of these measures is widely used as it enables quantities to be referred to a fixed reference configuration, facilitating the manipulation of integrals over material bodies. The second, which is also called the Jaumann stress by some authors [KOITER (1975), CHRISTOFFERSEN (1973) and DILL (1974)], has been used by HILL (1975) and BIOT (1965), and more recently it has been used extensively by OGDEN (1977).

The three measures, Nominal, Biot and Cauchy introduced above, are related as follows:-

$$\underline{\sigma} = J^{-1} \underline{\alpha}\, \underline{S} \qquad \text{(a)}$$
$$\underline{s} = \tfrac{1}{2}\, (\underline{S}\underline{R} + \underline{R}^T\underline{S}) \qquad \text{(b)} \qquad 1.3.2$$

J and $\underline{R}$ are as defined in equations 1.2.4 and 1.2.5.

Corresponding to 1.3.1 there is a relationship for $\underline{S}$,

$$T_i = S_{\mu i}\, N_\mu \qquad , \qquad 1.3.3$$

where N_μ is the unit normal to the undeformed surface. This equation serves to define the nominal traction T_i which is interpretted as the force per unit undeformed area. This interpretation is easily confirmed using Nanson's formula, which relates surface areas. If $d\underline{\Sigma}$ is a surface element in the undeformed configuration and $d\underline{S}$ the corresponding surface after a deformation 1.2.1, then

$$d\underline{S} = J\underline{\beta}\, d\underline{\Sigma} \qquad , \qquad 1.3.4$$

where $\underline{\beta}$ is the inverse of $\underline{\alpha}$. Employing the definition of the t_i of equation 1.3.1 with 1.3.1, 1.3.2(a) and 1.3.4 allows the interpretation of T_i given above to be recovered.

SECTION 1.4 CONSTITUTIVE RELATIONS: ELASTICITY

In this thesis attention is confined to Green (or Hyper) elastic materials. Standard manipulations, see OGDEN [1975(a)], yield the following expression:-

$$\oint S_{\mu i}\, d\alpha_{i\mu} \qquad 1.4.1$$

for the energy required in taking a material element through a closed cycle in strain space. Equating this expression to zero, as is conventionally done for elastic materials, yields the conclusion that $\underline{S}$ is a conservative field. The potential for $\underline{S}$ is the Strain Energy Density function which we denote by W. An elastic material is said to be HYPERELASTIC if it is assumed that it possesses a strain energy density function.

The function W may be regarded as a function of any strain measure. Using this characteristic, and the property of being a potential for the stress field, stress measures may be defined for any selected strain measure. The stress measures so defined are said to be conjugate, in the sense of HILL [1968(a)(b)], to their corresponding strain measure. Thus, for example, if the strain measures of 1.2.3 are taken, then their companion (conjugate) stress tensors $\underline{\sigma}^{(m)}$, may be defined as

$$dW = \underline{\sigma}^{(m)}\, d\underline{e}^{(m)}$$

or 1.4.2

$$\underline{\sigma}^{(m)} = \partial W / \partial \underline{e}^{(m)} \quad .$$

In particular, $\underline{S}$, the nominal stress, is conjugate to $\underline{\alpha}$, the deformation gradient, with

$$S_{\mu i} = \partial W / \partial \alpha_{i\mu} \quad , \qquad 1.4.3$$

and $\underline{t}$, the Biot stress, is conjugate to $\underline{U}$, the right Green strain measure, with

$$t_{\mu v} = \partial W / \partial u_{\mu v} \quad . \qquad 1.4.4$$

The relationships 1.4.3 and 1.4.4, in that they relate stress and strain measures, are constitutive laws, indeed, for a given W they are equivalent descriptions of the material. It is to be noted that the conventional Cauchy stress $\underline{\sigma}$ is not, in general, conjugate to any strain measure.

The strain energy density function W may be taken as defining the material. The form of W is, however, not arbitrary but must satisfy certain conditions so that the defined material will be physically reasonable. The problem of just what restrictions and inequalities should be imposed on W is one of great significance in contemporary continium mechanics and is at present unresolved.

The local elastic state of a material element must be independent of any local rigid rotation, that is, independent of the $\underline{R}$ of equation 1.2.5. This requirement is a statement of the Principle of Material Objectivity in its simplest form, and is fully discussed in TRUESDELL & NOLL (1965), section 19A. The requirement may well be incorporated in the strain measure adopted such as those of 1.2.3, or alternatively, it may be incorporated in the material description W, which must be constrained as

$$W(\underline{\alpha}) \equiv W(\underline{P}\underline{\alpha}) \qquad \forall\ \underline{Q}\ \text{s.t.}\ \underline{P}\underline{P}^T = \underline{I} \qquad 1.4.5$$

which is equivalent to

$$W(\underline{\alpha}) \equiv W(\underline{U}), \qquad 1.4.6$$

where $\underline{I}$ is the identity tensor and $\underline{U}$ right stretch tensor of 1.2.5.

Another requirement to be imposed on W is that it must incorporate any material symmetrics of the material under consideration. This means that W must be insensitive to rotations $\underline{Q}$, of the Lagrangian frame, where $\underline{Q}$ is contained in the isotropy group. It is assumed here that the material is fully isotropic and the consequential restriction on W is thus

$$W(\underline{\alpha}) \equiv W(\underline{\alpha}\underline{Q}) \qquad \forall\ \underline{Q}\ \text{s.t.}\ \underline{Q}\underline{Q}^T = \underline{I} \qquad 1.4.7$$

which is equivalent to

$$W(\underline{\alpha}) \equiv W(\underline{V}), \qquad 1.4.8$$

$\underline{V}$ being the left stretch tensor of 1.2.5

Now 1.4.5 and 1.4.7 may be combined in a natural way to yield the relationship

$$W(\underline{\alpha}) \equiv W(\underline{P}\ \underline{\alpha}\ \underline{Q}^T),\ \forall\ \underline{P}\underline{P}^T = \underline{I} = \underline{Q}\underline{Q}^T, \qquad 1.4.9$$

for all fully isotropic materials. A consequence of this result is that W may be considered to be a function of only 3 independent scalar variables. That is

$$W \equiv W\ (p,\ q,\ r), \qquad 1.4.10$$

where p, q and r are three linearly independent functions of the (eigen) principal values of $\underline{U}$ (or $\underline{\alpha}^T\underline{\alpha}$). In particular, they may be the principal similarity invariants, or indeed, the principal values. W must be symmetric as a function of the principal values.

SECTION 1.5 PHYSICAL PRINCIPLES: MECHANICS

The formation developed so far allows the elastostatic state of a material element to be described. In order that the states are properly selected, governing equations must be employed. These governing equations are developed as a consequence of simple physical and geometrical constraints. The laws and their interpretted forms are briefly outlined below but further discussions of these may be found in any of the standard texts, JAUNZEMIS (1967) for example

(i) Principle of Material Impenetrability:

$$J = \det \underset{\sim}{\alpha} > 0 \qquad 1.5.1$$

everywhere within the material. This not only ensures the invertability of the deformation map 1.2.1, but ensures that no two material points occupy the same location.

(ii) Conservation of Mass:

$$\rho_o = J\rho \qquad 1.5.2$$

where ρ_o and ρ are the mass densities in the reference and deformed states respectively, and J is defined by 1.2.4. It is this principle which necessitates the strictly positive condition in 1.5.1 and 1.2.4

(iii) Conservation of Linear Momentum:

$$\sigma_{ji,j} + \rho f_i = D^2x_i/Dt^2 \qquad 1.5.3$$

where f_j represents the body force per unit undeformed volume. In the context of elastostatics when no body forces are admitted, this condition reduces to

$$\sigma_{ij,i} = 0$$

or 1.5.4

$$S_{\mu i,\mu} = 0 .$$

In this latter form the expressions are conventionally referred to as the Equilibrium Equations.

(iv) Conservation of Angular Momentum

$$\sigma_{ij} = \sigma_{ji}$$

or 1.5.5

$$\alpha_{ik}S_{kj} = S_{jk}\alpha_{ki},$$

the latter of these being a consequence of 1.3.2(a), given the former

(v) Strain Compatibility

$$\alpha_{i\mu,v} = \alpha_{iv,\mu} \qquad 1.5.6$$

This condition ensures that the matrix of deformation gradients is derivable from a deformation field, that is, that equation 1.2.3 may be solved for a deformation field $\underline{x}(\underline{X})$.

The five conditions given above are necessary and sufficient to ensure that any solution pair ($\underline{S}$, $\underline{\alpha}$) is mathematically admissible.

SECTION 1.6 THE COMPLETE BOUNDARY-VALUE PROBLEM

It has been emphasised that a variety of variables may be adopted in describing a problem. Which variables are chosen is often a matter of personal preference, but it is conventionally a compromise between ease of interpretation, ease of manipulation and suitability for the particular problem in hand. For the purpose of this thesis a Lagrangian formulation is adopted, all quantities being referred to the undeformed configuration. The conjugate pair ($\underline{S}$, $\underline{\alpha}$) of 1.4.3 is employed almost exclusively, although the pair ($\underline{t}$, $\underline{U}$) as in 1.4.4 also play an important role.

In addition to the theory and results discussed in the previous sections, all that now remains is to consider the boundary conditions before a problem in elastostatics may be posed. The basic governing equations of elastostatics are demonstrably of second order and for a well-defined non-singular problem, elliptical in character. It is thus necessary to specify conditions on all points of the boundary in order for a problem to be well posed. The boundary conditions may either be one of place, where a displacement is specified, or one of stress, where an applied traction is specified. Given a material volume V, in the undeformed state, with boundary Σ, with the displacements $\hat{x}_i$ specified on part Σ_x, of the boundary, and with nominal tractions $\hat{t}_i$ on Σ_t, where $\Sigma = \Sigma_x \cup \Sigma_t$, a complete problem may be stated as:-

Find the pair ($\underline{S}$, $\underline{\alpha}$) satisfying the following

$$S_{\mu i,\mu} = 0 \qquad \text{in } V, \quad \text{(a)}$$

$$\alpha_{ik,l} = \alpha_{il,k} \quad \text{in } V \qquad \text{(b)}$$

with 1.6.1

$$S_{\mu i}N_\mu = \hat{t}_i \qquad \text{on } \Sigma_t \quad \text{(c)}$$

and

$$x_i = \hat{x}_i \qquad \text{on } \Sigma_x \quad \text{(d)}$$

where N_μ is the outward unit normal to Σ_t. The equations $\alpha_{i\mu} = x_{i,\mu}$, $S_{\mu i} = \partial W/\partial \alpha_{i\mu}$ and $\Sigma = \Sigma_x \cup \Sigma_t$, are also relevant. The strain energy function W is taken as defining the material to be considered.

The remainder of this thesis is dedicated to the investigation of the system 1.6.1 and its solution. Assumptions will be made regarding isotropy and the dimensionality of the deformation. Those are shown to lead to simplification of the system and will finally lead to a formal theoretical and analytical solution.

SECTION 1.7 A COMPLEMENTARY FORMULATION

A formal solution to the system 1.6.1 may be obtained by adopting the deformation field $x_i(X_\mu)$ as a primative, as the strain compatability equations 1.6.1(b) are then automatically satisfied. If the stress field is represented in terms of the deformation gradient via the constitutive law 1.4.3, for a given W, then it may be employed to eliminate the stress field from the equilibrium equations 1.6.1(a) and the stress boundary conditions 1.6.1(c). The resulting partial differential system in x_i is of second order.

Conversely, the stress potentials may be introduced such that the equilibrium equations 1.6.1(a) are satisfied automatically. This postulation or introduction of stress potentials is a widely used technique especially when planar deformations are to be considered. Airy's stress potential is, in fact, a potential for the stress components of a planar stress field, see MUSKHELISHVILI (1963) or Appendix 2. In developing this inverse method as in the paragraph above, an inverse constitutive law is required. The inverse law allows the deformation gradient to be expressed as a function of the stress potentials, so as to eliminate the former from 1.6.1(b) and 1.6.1(c). In other words, an expression of the form

$$\alpha_{i\mu} \equiv \alpha_{i\mu}\,(S_{vj}) \qquad 1.7.1$$

is required. The question of the existence and validity of this inversion has, until recently, not been satisfactorily answered as uniqueness is not guaranteed and the results also depend upon the choice of conjugate variables.

Clarification of the situation has been provided recently in a paper by OGDEN (1977). Given $W(\underline{\alpha})$, the strain energy density function, then its Legendre dual $Wc(\underline{S})$, the complementary energy density function may be derived as follows

$$Wc(S_{\mu i}) = S_{\mu i}\alpha_{i\mu} - W(\alpha_{i\mu}) \quad . \qquad 1.7.2$$

This expression may be formally differentiated with respect to the strain measure allowing equation 1.4.3 to be recovered. Differentiating it with respect to the stress, an inverse relationship is obtained,

$$\alpha_{j\mu} = \partial Wc/\partial S_{\mu j} \quad . \qquad 1.7.3$$

Thus the question of inversion of the constitutive law to yield 1.7.1 is equivalent to the question of the existence of the potential Wc, as a function of $\underline{S}$.

Employing the conjugate pair $(\underline{t}_1, \underline{U})$, Ogden has demonstrated that the inversion, 1.7.1, exists in some neighbourhood of the origin in strain space and is locally unique provided this neighbourhood is convex. A further discussion of the complementary constitutive law 1.7.3, when planar deformations only are considered, may be found in ISHERWOOD & OGDEN [1977 (a)].

The duality of the system 1.6.1 as regards the interchange of the deformation field and stress potentials may be further exploited, as has been pointed out by HILL & SHIELD (1974) and OGDEN [1975(b)]. These authors, employing this duality, have concluded that the stress potentials may be viewed as the deformation for a different boundary data/material pair. The deformation field may correspondingly be viewed as the stress potentials for that same pair. The dual boundary conditions, as they are referred to, have a non-trivial interpretation but the dual material is simply determined from W by taking the Legendre transform as in 1.7.2, where Wc defines the new material.

Throughout this thesis the topic of duality, and the complementary formulation, will be repeatedly discussed as it seems to be a most resiliant feature of the system 1.6.1 and its solutions.

CHAPTER 2: PLANE STRAIN

SECTION 2.1 INTRODUCTION

In this chapter attention is restricted to the consideration of a particular deformation class, that of plane strain. The equations of elasticity of Chapter 1 will be simplified and reduced to a form amenable to analytic solution. The solution of these equations for a particular problem class will be presented.

In linear (classical) elasticity, the technique of superposition is widely employed in the solution of many problems. Additionally, assumptions are made which aid practicability of solution and also ease interpretation and understanding of that solution.

When a non-linear constitutive law is employed and/or finite deformations admitted, then superposition of solution fields is precluded. This does not apply when questions of infinitesimal stability are being addressed, ie where virtual infinitesimal displacements are considered. In this case superposition may be employed, see WU & WIDERA (1969), KERR & TANG (1962) and SENSENIG (1964) for examples.

When the solution of a fully non-linear elastostatic problem is attempted, the number of difficulties arising is considerable, as is evidenced by the relatively few solutions presented in the literature. Not the least of these difficulties is the problem of dimensionality. The dimensionality of a problem is not simply that of the physical space in which it is posed, but is a measure of the irreducible number of independent variables involved in the formulation. It is usual for assumptions to be made in order to reduce the dimensionality of the problem. Often employed assumptions are those of isotropy and homogeneity which remove both spatial and directional variations in the material problems.

The assumption of inextensibility in one or two directions is also employed, especially since the introduction of carbon fibre materials, see PIPKIN (1975) and GREEN & ZERNA (1954). Another assumption is that of incompressibility. This is specified as

$$J = 1 \quad \text{everywhere,} \tag{2.1.1}$$

where J, of equation 1.2.2, is the Jacobian of the deformation. The form 2.1.1 is widely used.

The assumption of incompressibility will not be adopted in this thesis. Indeed, the question of volume changes is discussed in some considerable detail.

A more direct way of reducing the dimensionality of the problem is to reduce the dimension of the physical space considered. A first step would be to consider problems which are independent of one spatial co-ordinate. These assumptions are adopted throughout this thesis. The problems considered are those of two-dimensional elastostatics.

Problems have been posed, and solved, in diverse two-dimensional subspaces. Convected co-ordinates are usually employed. GREEN & ZERNA (1954), GREEN & ADKINS (1960), STICKFORTH (1975), ADKINS, GREEN and NICHOLAS (1954) and the many texts on shell theory are examples of the use of this assumption. FREUND (1972), SIH (1973), CRAGGS (1960) and BROBERG (1967), in considering crack problems, illustrate the effective use of flat two-dimensional subspaces. In this thesis the two-dimensional subspace considered is flat and is spanned by two rectangular Cartesian co-ordinates.

The consideration of a flat, two-dimensional subspace corresponds, physically, to the consideration of prismatic material configurations, the generators of the prism being perpendicular to the subspace. Boundary conditions are assumed constant along any generator. This situation is generally referred to as that of PLANE STRAIN. It must be distinguished from the situation of PLANE STRESS which deals with plates of materials, and not prisms, see WU & WIDERA (1969) for an example. In classical elasticit theory the two situations are functionally identical but the material constants are different.

The restriction of the region of interest to that of a flat, two-dimensional subspace has two further advantages, apart from simply reducing the number of independent variables:-

(a) It enables the formidable armoury of complex variable theory to be used. This has, to date, primarily been done within the confines of classical elasticity. The approach of MUSKHELISHVILI is a good example, see Appendix 2.

(b) Solution fields may immediately be compared with experimental observations, this being particularly practicable when plane stress is considered. Graphical representation is also made possible.

Of the two advantages, (a) is used to great advantage in Chapters 4 and 5, while (b) is employed only insofar as graphical results are presented in Chapter 6.

It is not unusual for authors to use combinations of the assumptions mentioned above. KLINGBEIL & SHIELD (1966) consider planar deformations of incompressible materials. ANTMAN (1976) introduces a series of papers dealing with the ordinary differential equations of elastic beam theory.

The final assumption mentioned here is that of symmetry. This assumption is often used to simplify problems and will be used when deriving illustrative solutions in this thesis. It is used extensively in ISHERWOOD (1976). When combined with those assumptions above, symmetry facilitates the solution of many problems and is widely used.

The first part of this chapter consists of a statement of the field equations and boundary conditions of elastostatics, suitably restricted to plane strain. This will be followed by a discussion of admissible forms for the constitutive law. Allied with this, a discussion of the admissible forms of the material description - namely the strain energy density function is given. Several deformation invariants (see 1.4.8) will be introduced and discussed. Lastly, symmetry arguments will be employed to solve a variety of problems for various materials.

SECTION 2.2 REDUCED FIELD EQUATIONS

Attention is henceforth restricted to the deformation of a flat, two-dimensional subspace. For manipulative convenience the $X_3 (=x_3)$ is taken to be perpendicular to this subspace. All deformations considered are constrained to be such that $x_3 = X_3$. All quantities introduced in Chapter 1 will be considered to be functions of (X_1, X_2) or (x_1, x_2) only. Variables will be suitably restricted to the dimensionality of the subspace. In particular, the component representations of tensors will be 2 x 2 matrices. Summation will henceforth be over the values 1 and 2 only.

The reduction in dimensionality allows the governing equations of 1.6.1 to be written in full:-

$$\begin{aligned} S_{11,1} + S_{21,2} &= 0 \\ S_{12,1} + S_{22,2} &= 0 \end{aligned} \qquad 2.2.1$$

are the equilibrium equations 1.6.1(a), in terms of the nominal stress (with no body forces).

$$\begin{aligned} \alpha_{12,1} - \alpha_{11,2} &= 0 \\ \alpha_{22,1} - \alpha_{21,2} &= 0 \end{aligned} \qquad 2.2.2$$

are the deformation compatibility equations of 1.6.1(b). The similarity of the two pairs of equations has been pointed out in Chapter 1. This is again considered in Section 2.4. The traction - stress relationship may also be written in full as

$$\begin{aligned} t_1^* &= S_{11}N_1 + S_{21}N_2 \\ t_2^* &= S_{12}N_1 + S_{22}N_2 \end{aligned} \qquad 2.2.3$$

where t_i^* represents the force per unit initial area. In the context of planar deformation, area is now to be interpretted as line length. N_μ is the outward normal to the undeformed surface (curve).

SECTION 2.3 ISOTROPIC STRAIN ENERGY DENSITY FUNCTION: INVARIANTS CONSTITUTIVE LAW

In Section 1.4 it was shown that for isotropic materials W, the strain energy density function is a function of the similarity invariants of $\underset{\sim}{\alpha}^T\underset{\sim}{\alpha}$, $\underset{\sim}{\alpha}$ being the deformation gradient of 1.2.2. The restriction to two spatial dimensions results in there being only two independent invariants. The conventional invariants are

$$I = \tfrac{1}{2}\left(\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{21}^2 + \alpha_{22}^2\right)$$
$$J \equiv \det(\underset{\sim}{\alpha}) = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \quad . \qquad 2.3.1$$

However, in this thesis it is found to be advantageous to employ the invariant pair

$$p \equiv \left\{2(I + J)\right\}^{\frac{1}{2}} = \left\{(\alpha_{11}+\alpha_{22})^2 + (\alpha_{12}-\alpha_{21})^2\right\} > 0$$
$$2.3.2(a)$$
$$q \equiv \left\{2(I - J)\right\}^{\frac{1}{2}} = \left\{(\alpha_{11}-\alpha_{22})^2 + (\alpha_{12}+\alpha_{21})^2\right\} \geqslant 0 \ .$$

In order to obtain a feel for the invariants p and q it is best to look at the form they assume on the principal axes of $\underset{\sim}{\alpha}^T\underset{\sim}{\alpha}$, i.e. when $\underset{\sim}{\alpha}^T\underset{\sim}{\alpha}$ is referred to its eigenvectors as a base, where it is diagonalized. Letting λ_i^2, i = 1,2 be the diagonal (principal or eigen) values; $\lambda_i > 0$ are called the principal stretches. Then 2.3.2(a) becomes

$$p = \lambda_1 + \lambda_2$$
$$2.3.2(b)$$
$$q = /\lambda_1 - \lambda_2/ \quad .$$

In this thesis conditions when q = 0 will subsequently be seen to be of importance. From 2.3.2(b) we note that this occurs when the principal stretches are equal. This condition is associated with a hydrostatic stress, where the principal stresses are also equal.

Now, if $\underset{\sim}{e}^{(m)}$ is any strain measure of 1.2.6 and $\underset{\sim}{\sigma}^{(m)}$ its conjugate (see Section 1.4) stress measure, then for isotropic materials their principal directions coincide; see HILL (1968a) for proof. The conditions q = 0, being associated with equal principal stresses (strains), leaves the principal directions indeterminate. In this sense such a condition is singular.

It was pointed out in Section 1.4 that any rational symmetric functions of the principal stretches suffice as invariants. A further pair of invariants to be employed is

$$p^* = p + q, \quad q^* = p - q \quad . \qquad 2.3.3$$

In Section 1.3 the constitutive law was determined for a given W as

$$S_{\mu i} = \partial W / \partial \alpha_{i\mu} \quad . \qquad 2.3.4$$

Introducing the invariant pair p and q of 2.3.2 into 2.3.4 produces

$$S_{\mu i} = W_p \frac{\partial p}{\partial \alpha_{i\mu}} + W_q \frac{\partial q}{\partial \alpha_{i\mu}} \quad . \qquad 2.3.5$$

Applying the polar decomposition theorem of 1.2.3 to $\underline{\alpha}$ yields

$$\underline{\alpha} = \underline{V}\,\underline{R} \qquad 2.3.6$$

where

$$\underline{R}^T\underline{R} = \underline{I} \text{ and } \underline{V} = \underline{V}^T \quad . \qquad 2.3.7$$

Then writing

$$\underline{R} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \quad , \qquad 2.3.8$$

equation 2.3.6 may be used to obtain the result

$$\tan\chi = \frac{\alpha_{21} - \alpha_{12}}{\alpha_{11} + \alpha_{22}} \quad , \qquad 2.3.9$$

as a necessary condition that 2.3.7 (ii) is satisfied. Additionally, using 2.3.6 and 2.3.2 with 2.3.7, the following may be derived

$$\left.\begin{aligned} \alpha_{11} + \alpha_{22} &= p\cos\chi \\ \alpha_{21} - \alpha_{12} &= p\sin\chi \quad . \end{aligned}\right\} \qquad 2.3.10$$

Formal differentiation of the invariant pair (p, q) with respect to the deformation gradient results in

$$\frac{\partial p}{\partial \alpha_{i\mu}} = \frac{\alpha_{i\mu} + J\mathcal{C}_{\mu i}}{p}$$

and 2.3.11

$$\frac{\partial q}{\partial \alpha_{i\mu}} = \frac{\alpha_{i\mu} - J\mathcal{C}_{\mu i}}{q} \quad ,$$

where $\underline{\mathcal{C}}$ is the inverse $\underline{\alpha}$. The results 2.3.10(1) and 2.3.11(1) may be used in conjunction to obtain

$$\left[\frac{\partial p}{\partial \alpha_{i\mu}}\right] = R_{i\mu} \quad . \qquad 2.3.12$$

Prompted by 2.3.12, consider the derivative $\partial q/\partial \alpha_{i\mu}$. Simple algebra verifies the following

$$\frac{\partial q}{\partial \alpha_{11}} = -\frac{\partial q}{\partial \alpha_{22}} , \quad \frac{\partial q}{\partial \alpha_{12}} = \frac{\partial q}{\partial \alpha_{21}}$$

and

$$\left(\frac{\partial q}{\partial \alpha_{11}} \right)^2 + \left(\frac{\partial q}{\partial \alpha_{12}} \right)^2 = 1 \quad .$$

These results suggest that a further angle χ^* and its associated rotation (improper) $\underline{R}^*$ be introduced, such that

$$\left[\frac{\partial q}{\partial \alpha_{i\mu}} \right] = R^*_{i\mu} = \begin{bmatrix} \cos\chi^* & \sin\chi^* \\ \sin\chi^* & -\cos\chi^* \end{bmatrix} \qquad 2.3.13$$

and

$$\tan\chi^* = \frac{\alpha_{12} + \alpha_{21}}{\alpha_{11} - \alpha_{22}} \quad . \qquad 2.3.14$$

No easy interpretation of the angle χ^* can be made. If a further deformation were to be introduced, onto that already undergone, such that

$$x_1 \rightarrow x_1 = x_1'$$

and

$$x_2 \rightarrow -x_2 = x_2' \quad ,$$

then

$$\chi' = -\chi^*$$

and

$$\chi^{*\prime} = \chi \quad .$$

Where quantities with a superposed prime refer to the new configuration. In addition

$$\chi^* - \chi = 2\,\theta_E \quad . \qquad 2.3.15$$

The angle θ_E ($0 \leq \theta_E \leq \pi/2$) is the orientation of the principal axes of the left Green strain tensor $\underline{V}$ of 1.2.3, with respect to the background frame.

In summary, the constitutive law 2.3.5 may be written as

$$\underline{S} = W_p \underline{R} + W_q \underline{R}^* \qquad 2.3.16(i)$$

when 2.3.12 and 2.3.13 are employed.

It must be appreciated that the exact form of the constitutive law depends intimately on the invariants chosen. If the pair (I,J) were employed 2.3.16(i) becomes

$$\underline{S} = W_I \underline{\alpha}^T + J W_J \underline{\mathcal{E}} \qquad 2.3.16(ii)$$

where $\underline{\alpha}$ is again the deformation gradient and $\underline{\mathcal{E}}$ its inverse. There is no particular merit in the selection of any invariant pair, save their applicability to the problem in hand.

Consider a material class defined by

$$W = f(p) + \tfrac{1}{2}\mu q^2 , \qquad 2.3.17(i)$$

where μ is a constant and $f(p)$ some function whose behaviour need not be specified at present except that it be twice continuously differentiable. This 2.3.17(i) form may be re-arranged to yield

$$W = F(p) - \tfrac{1}{2}\mu (p^2 - q^2) \qquad 2.3.17(ii)$$

In this form it may be recognised as that considered in JOHN (1960) as the Harmonic material. This class of materials is of particular interest and will be used extensively. The second term of 2.3.17(ii) may, using 2.3.2(a), be written as

$$2\mu J \quad . \qquad 2.3.18$$

Now in deriving the constitutive law from 2.3.17(ii) with the second term replaced by 2.3.8, a term of the form

$$\frac{\partial J}{\partial \alpha_{i\mu}} = J \mathcal{E}_{i\mu} \qquad 2.3.19$$

results. Writing the strain compatibility equations 2.2.2 as

$$(J \mathcal{E}_{i\mu}),_{\mu} = 0 \qquad 2.3.20$$

it may be noted that the term $2\mu J$ does not contribute to the equilibrium equations 2.2.1

SECTION 2.4 DUALITY: THE COMPLEMENTARY ENERGY FUNCTION

In Section 1.7 an alternative material definition, that of the Complementary Energy Function Wc, was introduced. This function is either explicity defined or may be obtained from W via a Legendre contact transformation. In this section the properties of this function and a duality in the formalism are discussed. This duality will be a recurrent theme throughout this thesis.

The function Wc is assumed to exist and be twice differentiable in a simply connected neighbourhood of the origin in stress space. If Wc is determined from W this neighbourhood is determined by the condition $\det [\partial \underline{t}/\partial \underline{u}] > 0$; see Section 1.7 for details.

As in Section 1.4, the principles of material frame indifference and isotropy may be invoked with respect to Wc to yield

$$Wc \equiv Wc(p_T, q_T) \quad . \qquad 2.4.1$$

This result is analogous to 1.4.8 with

$$p_T = \pm\sqrt{2}(I_T+J_T)^{\frac{1}{2}} = \pm\left\{ (S_{11}+S_{22})^2 + (S_{21}-S_{12})^2 \right\}^{\frac{1}{2}} ,$$
$$q_T = \sqrt{2}(I_T-J_T)^{\frac{1}{2}} = \left\{ (S_{11}-S_{22})^2 + (S_{21}+S_{12})^2 \right\}^{\frac{1}{2}} \qquad 2.4.2$$

and

$$I_T = \tfrac{1}{2}S_{\alpha\epsilon}S_{\alpha\epsilon} \text{ with } J_T = S_{11}S_{22} - S_{12}S_{21} \quad . \qquad 2.4.3$$

These are four similarity invariants of the nominal stress. Substituting for $\underline{S}$ from 2.3.17 in 2.4.2 a little, but tedious, algebra serves to demonstrate that

$$W_p = \frac{p_T}{2} \text{ and } W_q = \frac{q_T}{2} \quad . \qquad 2.4.4$$

In order to expose the algebraic duality of the system of equations being considered, it must be noted that Wc is obtained from W via the transform

$$W + Wc = \alpha_{i\mu}\, S_{\mu i} \quad . \qquad 2.4.5$$

A consequence of this is that

$$\alpha_{i\mu} = \frac{\partial Wc}{\partial S_{\mu i}} ,$$

and hence, employing 2.4.1,

$$\alpha_{i\mu} = Wc_{p_T}\frac{\partial p_T}{\partial S_{\mu i}} + Wc_{q_T}\frac{\partial q_T}{\partial S_{\mu i}} \quad . \qquad 2.4.6$$

From 2.4.3 a little algebra confirms the following

$$\frac{\partial p_T}{\partial S_{11}} = \frac{\partial p_T}{\partial S_{22}} \; , \; \frac{\partial p_T}{\partial S_{12}} = \frac{-\partial p_T}{\partial S_{21}}$$

with 2.4.7

$$\left(\frac{\partial p_T}{\partial S_{11}} \right)^2 + \left(\frac{\partial p_T}{\partial S_{12}} \right)^2 = 1 \quad .$$

The following may also be verified

$$\frac{\partial q_T}{\partial S_{11}} = \frac{-\partial q_T}{\partial S_{22}} \; , \; \frac{\partial q_T}{\partial S_{12}} = \frac{\partial q_T}{\partial S_{21}}$$

with 2.4.8

$$\left(\frac{\partial q_T}{\partial S_{11}} \right)^2 + \left(\frac{\partial q_T}{\partial S_{12}} \right)^2 = 1 \quad .$$

The results 2.4.7 and 2.4.8 prompt the introduction of two angles χ_T and χ_T^* say, such that

$$\mathrm{Tan}\chi_T = \frac{S_{12}-S_{21}}{S_{11}+S_{22}} = \frac{\alpha_{21}-\alpha_{12}}{\alpha_{11}+\alpha_{22}} \qquad 2.4.9$$

and

$$\mathrm{Tan}\chi_T^* = \frac{S_{12}+S_{21}}{S_{11}-S_{22}} = \frac{\alpha_{21}+\alpha_{12}}{\alpha_{11}-\alpha_{22}} \quad . \qquad 2.4.10$$

The second equality in both of the above, being determined from equation 2.4.6, affords comparison with the previously defined angles χ and χ^*. On comparing 2.4.9(2) and 2.4.10(2) with 2.3.9 and 2.3.10 it may be concluded that

$$\chi_T = \chi \text{ and } \chi_T^* = \chi^* \quad . \qquad 2.4.11$$

In summary, the following may be written

$$\underline{\alpha} = Wc_{p_T} \underline{R} + Wc_{q_T} \underline{R}^* \qquad (a)$$

with 2.4.12

$$\underline{S} = \frac{p_T}{2} \underline{R} + \frac{q_T}{2} \underline{R}^* \qquad (b)$$

or

$$\underline{\alpha} = \frac{p}{2} \underline{R} + \frac{q}{2} \underline{R}^* \qquad (a)$$

with 2.4.13

$$\underline{S} = W_p\underline{R} + W_q\underline{R}^* \quad , \qquad (b)$$

R and R* are as defined in equations 2.3.8 and 2.3.13 respectively. Additionally, the psuedo-Legendre contact transform

$$Wc + W = \tfrac{1}{2}\,(pp_T + qq_T) \qquad 2.4.14$$

may be noted. Thus the duality and algebraic similarity of equations 2.2.1 and 2.2.2 are seen to carry over to the constitutive and inverse constitutive laws of 2.4.12 and 2.4.13.

The algebraic duality illustrated above is largely fortuitous insofar as it's clear form is dependent upon the choice of invariant pairs. Consider, for example, the further pair.

$$f \equiv f(p,q)\ ,\ g = g(p,q)\ , \qquad 2.4.15$$

where f and g will be assumed to be linearly independent. Then

$$W_p = W_f f_p + W_g g_p = p_T/2$$

and

$$W_q = W_f f_q + W_g g_q = q_T/2$$

where 2.4.4 has been employed. Then, provided f and g are linearly independent,

$$W_f = \tfrac{1}{2}\,(p_T g_p + g_q q_T)/(f_p g_q - f_q g_p)$$

with 2.4.16

$$W_g = \tfrac{1}{2}\,(f_p q_T - p_T f_q)/f_p g_q - f_q g_p)\ .$$

Consequently expressions of the form

$$W_f \equiv W_f(p_T, q_T)$$

and

$$W_g \equiv W_g(p_T, q_T)$$

may be written if and only if it is possible to invert W to obtain Wc. In which case

$$\frac{p}{2} = Wc_{p_T} \text{ and } \frac{q}{2} = Wc_{q_T}$$

may be employed to eliminate p and q from the expressions 2.4.16. It is only when f and g are linear functions of p and q that the simple algebraic duality is observed.

It should be noted that the expressions 2.4.9 and 2.4.10 for $\tan\chi$ and $\tan\chi^*$ serve to illustrate a universal relationship, that of

$$\frac{\alpha_{12}-\alpha_{21}}{\alpha_{11}+\alpha_{22}} = \frac{S_{21}-S_{12}}{S_{11}+S_{22}}$$

with

$$\frac{\alpha_{11}-\alpha_{22}}{\alpha_{12}+\alpha_{21}} = \frac{S_{11}-S_{12}}{S_{12}+S_{21}} \quad .$$

Notwithstanding the above discussion regarding fortuitous algebraic duality, there is a quite general duality to be noted. Indeed this duality is independent of the material isotropy, but it is restricted to plane strain. The duality is real, insofar as when the solution to problem I is known, then suitable interpretation of the variables results in the solution of another problem II, say. In OGDEN(1975b) the following is proved:-

If $(\underset{\sim}{\alpha},\underset{\sim}{S})$ is the solution pair for problem I, with boundary conditions

$$\hat{t}_i \text{ specified on } \mathcal{E}_r,$$

and

$$x_i \text{ specified on } \mathcal{E}_u,$$

with $\mathcal{E}_r \cup \mathcal{E}_u = \mathcal{E}$, the whole boundary, then (adjugate $\underset{\sim}{S}$, adjugate $\underset{\sim}{\alpha}$) is the solution pair for a problem II. If the material for the problem I is defined by a given W, then the material for problem II is that given by Wc, when the arguments are suitably interpreted as invariants of the new deformation gradient. The duality of the boundary data is not as straightforward. It may be summarised as follows:-

> "Thus, if the position (or displacement) of the boundary is specified for the original problem it follows that the tangential components $t_\mu \alpha_{i\mu}$ of the deformation gradient on the boundary can be calculated. This immediately specifies the components of traction $n_\mu S_{\mu i}$ for the dual problem. Conversely if the traction is specified in the original problem the tangential component of $\underset{\sim}{\alpha}$ on the boundary is known for the dual problem." OGDEN (1975b, p.88)

(t_μ and n_μ are simply the tangential and normal vectors to the material surface).

This completes the formal discussion of duality. However, it will be referred to again in Chapter 3, where it figures in a new formal approach to the solution of problems developed there. At present the duality has the status of an elegant formal relationship and as such is a satisfying piece of mathematics. As far as the author is aware it has not been employed in the solution of any problems.

SECTION 2.5 PHYSICALLY ACCEPTABLE CONSTITUTIVE LAW

It is not easy to resolve the question of what restrictions ought to be placed on a material description so that physically acceptable solutions can be obtained. Even within the somewhat reduced field of consideration, ie that of isotropic, homogeneous materials subjected to static plane strain deformations only, the consideration can only be fragmentary.

Consider a material defined by

$$W(p,q) \qquad 2.5.1$$

and, in particular,

$$W = f(p) + g(q) \quad . \qquad 2.5.2$$

In HILL [1957(a)] it is shown that for the BIOT stress measure with the current configuration as reference, that

$$\left[\frac{\partial \mathcal{L}}{\partial \underline{U}}\right] = \left[\frac{\partial^2 W}{\partial \underline{U} \partial \underline{U}}\right] \text{ +ve definite} \qquad 2.5.3$$

implies incremental uniqueness of solution and hence stability. The $\underline{U}$ in 2.5.3 is the RIGHT GREEN strain measure of 1.2.3 and is conjugate to (in the sense of 1.4.3) $\underline{\mathcal{L}}$. This result and some restrictions are discussed further in HILL (1968a). Denoting the principal BIOT stresses by $\mathcal{L}_i$, i=1,2 and recalling that the principal strains are denoted by λ_i, i=1,2, then 2.5.3 implies, and is implied by

$$\text{(a) } \det [\partial \mathcal{L}_i / \partial \lambda_i] \geqslant 0 \qquad 2.5.4$$

$$\text{(b) } (\mathcal{L}_1 - \mathcal{L}_2)(\lambda_1 - \lambda_2) > 0 \quad . \qquad 2.5.5$$

These conditions also ensure that a Wc may be determined from W. They are assumed to hold within some neighbourhood of the unstrained state in stress/strain space. OGDEN (1977) noted that if attention is focussed on a convex neighbourhood then 2.5.4 implies 2.5.5.

In terms of the assumed material descriptions (2.5.1 and 2.5.2), the restrictions 2.5.4 and 2.5.5, which are henceforth adopted, become

$$W_q \geqslant 0 \qquad 2.5.6$$

and

$$W_{pp}W_{qq} - W_{pq}^{\,2} > 0 \qquad 2.5.7$$

for form 2.5.1, and

$$g' \geqslant 0 \qquad 2.5.8$$

with

$$f''g'' > 0 \qquad 2.5.9$$

for 2.5.2. ()' denotes the derivative with respect to the argument.

To supplement conditions 2.5.6-9 others may be generated by subjecting the material to conceptual experiments. The results of these experiments are required to correspond qualitatively to intuition and result in further restrictions.

As a first example, consider a material body subjected to an applied hydrostatic pressure. To facilitate interpretation, the Cauchy stress is used as its associated tractions are measurable. Equation 1.3.2 may be used with 2.4.13(b) to yield

$$\sigma_{ij} = J^{-1} \left(W_p V_{ij} + W_q \frac{V_{ik} V_{kj}}{q} - \frac{J\, W_q}{q} \delta_{ij} \right) , \qquad 2.5.10$$

where δ_{ij} is again the kronecker delta and V_{ij} is the Left Green strain measure. Now $\underline{\sigma}$ and $\underline{V}$ are both symmetric and demonstrably coaxial. Referring to principal axes and equating diagonal elements, the following are obtained.

$$\sigma_1 = \frac{1}{\lambda_2} \left(W_p + W_q \ \mathrm{Sgn}\ (\lambda_1 - \lambda_2) \right)$$

$$\sigma_2 = \frac{1}{\lambda_1} \left(W_p - W_q \ \mathrm{Sgn}\ (\lambda_2 - \lambda_1) \right).$$

For the case of hydrostatic pressure when $\sigma_1 = \sigma_2 = \sigma$ say, and when $\lambda_1 = \lambda_2 = \lambda$ then the above reduce to

$$\sigma = \frac{1}{\lambda} \ W_p(2\lambda, 0) \ ,$$

Given this hydrostatic pressure λ is expected to be monotonic increasing as a function of σ, and to be equal to 1 when $\sigma = 0$. The first of these is guaranteed provided

$$\frac{d}{dp} \left(\frac{W_p(p,0)}{p} \right) > 0 \qquad 2.5.11(1)$$

for $p > 0$. The second is guaranteed by

$$W_p(2,0) = 0 \quad . \qquad 2.5.11(2)$$

Similarly, considering an applied uniaxial tension it may be concluded that

$$f'(p) \gtreqless 0 \text{ as } p \gtreqless 2 \; . \qquad 2.5.11(3)$$

A further reasonable requirement which is adopted here, is that the material definition must be asymptotically equal to the classical description, for small strains. In following JOHN (1960), consider a material defined by

$$W = F(p) + KJ,$$

as was introduced in Section 2.3. Then, proceeding as in JOHN (1960, page 249),

$$p-2 = (\lambda_1-1) + (\lambda_2-1) = \mathrm{tr}(\underline{V}-\underline{\delta})$$

$$J-1 = (\lambda_1-1) + (\lambda_2-1) + (\lambda_1-1)(\lambda_2-1)$$

$$= \mathrm{tr}\,(\underline{V}-\underline{\delta}) + \det\,(\underline{V}-\underline{\delta})$$

$$= \mathrm{tr}\,(\underline{V}-\underline{\delta}) + o[(\underline{V}-\underline{\delta})^2]$$

where tr denotes the trace and det the determinant. Writing $\underline{V}-\underline{\delta} = \underline{\eta}$, the above may be written as

$$p-2 = \mathrm{tr}\,\underline{\eta} \qquad 2.5.12$$

and

$$J-2 = \mathrm{tr}\,\underline{\eta} + o\,(\underline{\eta\eta}^T). \qquad 2.5.13$$

Expanding equation 2.5.10 about $\|\underline{\eta}\|$ as a small quantity then

$$\underline{\sigma} = [(W_p+W_J) + (W_{pp}+2W_{pJ}+W_{JJ}-W_p)\,\mathrm{tr}\,\underline{\eta}]\,\underline{\delta} + W_p\,\underline{\eta} + o\,(\underline{\eta\eta}^T), \qquad \mathbf{2.5.14}$$

is obtained with derivatives being evaluated at p = 2, J = 1, the undeformed configuration. Adopting a natural reference configuration, that is one which is both undeformed and unstressed, gives, from 2.5.14

$$W_p+W_J\Big|_{p=2,J=1} = 0 \qquad 2.5.15$$

Then setting

$$W_p\Big|_{p=2,J=1} = 2\mu \qquad 2.5.16$$

and

$$W_{pp}+2\,W_{pJ}+W_{JJ}\Big|_{p=2,J=1} = \lambda + 2\mu$$

equation 2.5.14 becomes

$$\underline{\sigma} = \lambda\underline{\delta}\,\mathrm{tr}\,\underline{\eta} + 2\mu\,\underline{\eta} + o\,(\underline{\eta\eta}^T) \qquad 2.5.17$$

λ and μ may be identified with the conventional Lamé constants. The restrictions normally placed on λ and μ may now be applied to a more general material definition via equations 2.5.15 and 2.5.16.

From the foregoing analysis a particular material may be identified as being of interest. It is that material which is simply an extension of the classical isotropic strain energy function, namely

$$W = \frac{\lambda + \mu}{2} (p-2)^2 + 2\mu (p-J-1) \qquad 2.5.18$$

or, in terms of the preferred invariants

$$W = \frac{\lambda + \mu}{2} (p-2)^2 + \frac{\mu}{2} q^2 \quad . \qquad 2.5.19$$

This material class is of sufficient importance to warrant a name and is conventionally termed the "SEMI-LINEAR" or "STANDARD" material. Applying conditions 2.5.6, 2.5.7 and 2.5.12 to this form for W, the following results:-

$$\mu q \geqslant 0 \ ,$$

$$\mu(\lambda + \mu) > 0$$

and

$$\frac{\lambda+\mu}{p^2} > 0$$

respectively. Together they imply

$$\mu > 0$$

and $\qquad 2.5.20$

$$\lambda + \mu > 0 \quad .$$

Finally, it should be noted that 2.5.15 and 2.5.16 may be cast with reference to W as a function of p and q as

$$W_p\Big|_{(2,0)} = 0 \quad , \quad W_q\Big|_{(2,0)} = 0$$

and

$$\frac{W_q}{q}\Big|_{(2,0)} = \mu \qquad 2.5.21$$

with

$$W_{pp}\Big|_{(2,0)} = \lambda + \mu .$$

From the discussion presented in this section it may readily be seen that the concept of a physically realistic material is ill-defined.

Restrictions may be obtained by a variety of means and, indeed, some will become apparent as solutions are generated in Chapter 4. In particular, one further restriction which will become evident in considering a pure shear is

$$f'(p) < \mu p \qquad 2.5.22$$

where f(p) is as in expression 2.3.17(1). It is included here for completeness, being generated in a similar fashion to 2.5.11(1)-(3). As a consequence of 2.5.22, it will be concluded that the semi-linear material description (2.5.19) is valid only for

$$p < 2\,\frac{(\lambda+\mu)}{(\lambda-\mu)}. \qquad 2.5.23$$

SECTION 2.6 RADIALLY SYMMETRIC PROBLEMS

In this section the theory developed in the preceding sections is employed in solving problems of a particular class for various materials. The problem class itself is distinguished only in that it may be solved, from first principles, for a variety of materials.

The problem class to be considered is characterised by the fact that the deformation is constrained to be radial (in the considered plane). The solution method employed is the ST. VENANT Semi Inverse.

With the notation of Section 1.2 the deformation may be described by

$$x_i = \rho X_i \qquad i = 1,2 \tag{2.6.1}$$

only. The radial expansion factor ρ is not constant. As a Lagrangian description is being used, ρ may be regarded as a function of the polar radius in the undeformed configuration, R.

$$R \equiv (X_\mu X_\mu)^{\frac{1}{2}} = r/\rho \ , \tag{2.6.2}$$

where r is the polar radius in the deformed configuration. Given equation 2.6.1 the deformation gradient may be determined as

$$\alpha_{i\mu} \equiv \frac{\partial x_i}{\partial X_\mu} = \rho\,\delta_{i\mu} + \frac{X_i X_\mu \rho'}{R}, \tag{2.6.3}$$

where ()' denotes differentiation with respect to the argument. In their turn the invariants p and q may be obtained by using 2.3.2 to yield

$$p = 2\rho + R\rho'$$

and (2.6.4)

$$q = \rho' R \quad .$$

Equations 2.3.8, 2.3.9, 2.3.12 and 2.3.14 allow the following to be derived

$$\text{Cos}\chi = \frac{\alpha_{11}+\alpha_{22}}{p} = 1, \ \text{Sin}\chi = 0$$

$$\text{Cos}\chi^* = \frac{\alpha_{11}-\alpha_{22}}{q} = \frac{X_1^2-X_2^2}{R^2} = \text{Cos}2\theta \ , \tag{2.6.5}$$

and

$$\text{Sin}\chi^* = \text{Sin}2\theta,$$

where θ is the angular co-ordinate of a material point in both the deformed and reference configurations. The above results could also be derived directly from the definition of the problem class with equation 2.3.16.

Applying the relationship 2.4.13(b) the nominal stress field, $\underset{\sim}{S}$ is determined as

$$\begin{aligned} S_{11} &= W_p + W_q \operatorname{Cos}2\theta \\ S_{12} &= S_{21} = W_q \operatorname{Sin}2\theta \\ S_{22} &= W_p - W_q \operatorname{Cos}2\theta \quad . \end{aligned} \qquad 2.6.6$$

As a direct consequence of the formulation, the strain compatibility equations 1.5.6 are satisfied directly. The equilibrium equation 1.5.4 remain to be satisfied. Now

$$S_{11,1}+S_{21,1} = 0 \Rightarrow W_{p,1}+(W_q \operatorname{Cos}2\theta)_{,1}+(W_q \operatorname{Sin}2\theta)_{,2} = 0$$

and

$$S_{12,1}+S_{22,2} = 0 \Rightarrow W_{p,2}+(W_q \operatorname{Cos}2\theta)_{,2}+(W_q \operatorname{Sin}2\theta)_{,1} = 0$$

are the reduced form of the equilibrium equations. These, after a few lines of basic but cumbersome manipulation result in the following form

$$\frac{d}{dR}(W_p+W_q) = \frac{-2W_q}{R} \quad . \qquad 2.6.7$$

The satisfaction of this equation is both necessary and sufficient for the equilibrium equations to hold. In terms of the complementary invariants as introduced by equation 2.4.2, equation 2.6.7 may be written

$$\frac{d}{dR}(p_T+q_T) = \frac{-2q_T}{R} \quad . \qquad 2.6.8$$

Equation 2.6.7 is seen to assume a simpler form if the invariants

$$f = p+q \text{ and } g = p-q \qquad 2.6.9$$

are employed, for then

$$W_p = W_f+W_g$$

with 2.6.10

$$W_q = W_f-W_g \quad .$$

Equation 2.6.7 then becomes

$$\frac{d}{dR}(RW_f) = W_g \quad . \qquad 2.6.11$$

It is to be noted that were the deformation gradient not derivable from a deformation field, then the strain compatibility equation would be reducible to the forms

$$\frac{d}{dR}(p-q) = \frac{2q}{R}$$

or 2.6.12

$$\frac{d}{dR}(Wc_{p_T} - Wc_{q_T}) = 2\frac{Wc_{q_T}}{R} .$$

In general, a problem of the defined class is solved by the substitution of a specific form for W into 2.6.7. The resulting equation will be a second order ordinary differential equation for $\rho(R)$. If the o.d.e. were to be of first order, it can be concluded that the material selected is not realistic; at least for problems of this class. In general, however, being of second order, two boundary conditions have to be specified in order that the solution be fully determined. These boundary conditions may be either that of displacement or of nominal traction specified at the surface of a cylinder; or exceptionally, as limiting conditions as R tends to infinity.

Considering now the traction boundary condition. The stress distribution of 2.6.6 may be rewritten as

$$S_{\mu i} = W_p\, \delta_{\mu i} + W_q\, [2\theta]_{\mu i} \qquad 2.6.13$$

where $[2\theta]$ is an improper rotation through angle 2θ, θ being defined above. The only tractions **preserving the symmetry required by the problem class are hydrostatic pressure or tension.** Applying 1.6.1(c) with normal ($\cos\theta$, $\sin\theta$), the following is obtained

$$S_R = W_p + W_q , \qquad 2.6.14$$

where S_R is the nominal pressure. The true traction is given by

$$t_R = (W_p + W_q)/\rho . \qquad 2.6.15$$

In terms of nominal pressure and considering the deformation of an annulus, the boundary conditions may be written as

$$S_R \equiv W_p + W_q = \tilde{S}_R$$

and **2.6.16**

$$\rho = \tilde{\rho}$$

where $\tilde{(\,)}$ denotes a specified value. One condition is specified on each of the boundaries.

Finally, before passing onto the solution of various problems it should be noted that 2.6.7 is linear in W. A consequence of this linearity is that:-

If a material $W^{(i)}$ has a general solution set $\Omega^{(i)}$ $i = 1,n$ of the equilibrium equations then the material $\sum_i a_i W^{(i)}$ has a solution set Ω, with

$$\bigcap_{i\in(1,n)} a_i\Omega^{(i)} \subset \Omega$$

The intersection may be null, this does not mean that there is no solution for such a material.

The detailed solution of a particular problem is presented here for illustrative purposes only. The material chosen is as in 2.3.13 but is adjoined by linear terms to ensure that the reference is both unstrained and unstressed. This normalisation eases comparison between the solutions produced for other materials; many such solutions are tabulated here. The other material definitions are also normalised. The modifications to the material definitions simply involve the addition of terms linear in p and q and a constant term, in order to ensure that

$$W(2,0) = 0$$

and 2.6.17

$$W_p(2,0) = W_q(2,0) = 0 \ .$$

The addition of these terms in no way affects the essential material behaviour as none figure in the equilibrium equation 2.6.7.

Rewriting 2.3.8 as

$$W = f(p) + B(p^2-q^2) \qquad 2.6.18$$

and modifying to ensure that 2.6.17 holds, leads to

$$W = (f(p)-f(2)) + f'(2)(2-p) + B(p^2-q^2) + 4Bp; \qquad 2.6.19$$

which defines the material to be considered. This, on inserting into 2.6.7 results in the reduced form of the equilibrium equations

$$\frac{d}{dR}\left(\frac{df}{dp}\right) = 0 \ . \qquad 2.6.20$$

This equation has two solutions

$$\text{(i)} \quad f(p) \propto p$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 2.6.21

$$\text{(ii)} \quad p \text{ constant}$$

The first of these, 2.6.21(i) is untenable as a reasonable solution as no restriction is placed on the deformation; the material so defined automatically satisfies 2.6.7. The second solution 2.6.21(ii) is of interest. Solving with 2.6.4(a) results in the following deformation class

$$\rho = C + DR^{-2} \; , \qquad 2.6.22$$

where C and D are constants. The expression for the invariant p is obtained from 2.6.4, and is

$$p = 2C.$$

The property that p is constant is seen to characterise many, but not all solutions presented in this thesis. Using 2.6.22 with 2.6.1(i) and 2.6.19 allows the nominal pressure S_R to be determined as

$$S_R = f'(2c) - f'(2) + 4BC - 4B + 4DBR^{-2}. \qquad 2.6.23$$

The expressions 2.6.22 and 2.6.23 allow boundary data as indicated by 2.6.16, to be incorporated to yield a complete solution. The full nominal stress field may then be recovered using 2.6.13. As an illustration:-

Consider a material configuration consisting of the entire Euclidean 3-space with a right circular cylinder radius 'a', and generators parallel to the X_3-axis removed. Assume that a nominal pressure P_a is applied to the interior of this cylinder. Then, adopting the assumption that

$$\rho \to 1 \text{ as } R \to \infty \; ,$$

may be concluded that

$$C = 1, \qquad 2.6.24$$

from 2.6.22. This, using 2.6.23 implies that

$$S_R = 4BDR^{-2}.$$

Equating this with P_a on R = a, D may be determined as

$$D = \frac{P_a a^2}{4B} \qquad 2.6.25$$

and consequently using 2.6.22 and 2.6.24,

$$\rho = 1 + \frac{P_a}{4B} \left(\frac{a}{R}\right)^2$$

with $\qquad$ 2.6.26

$$S_{\mu i} = P_a\left(\frac{a}{R}\right)^2 [2\Theta]_{\mu i},$$

where $[2\Theta]$ is as introduced in 2.6.6. The complete problem is thus solved.

It ought to be noted that both the deformation and stress fields determined above are independent of the particular form of f(p). This is not the normal situation. The constants C and D are to be expected to depend on f(p) via the boundary data. For this reason in the solutions tabulated below a particular f(p) will be adopted.

The three problems considered as representative of the class are:-

a) As considered above.

b) A finite annulus radii $a < b$ with zero displacement specified on $R = b$. A nominal pressure P_a applied on the inner surface $R = a$.

and c) The same configuration as b) but with pressures P_a and P_b applied to the inner and outer surfaces respectively.

Various materials are considered:-

<u>Material Class I</u>

$$W = Ap^2+B(p^2-q^2) - 4BP - 4AP \qquad 2.6.27$$

This corresponds to that discussed above with $f(p) \equiv Ap^2$

a) $\rho = 1 + \dfrac{P_a}{4B} a^2R^{-2}$, $\quad S_R = P_a a^2 R^{-2}$

b) $\rho = 1 + P_a[4A+4B(1-a^{-2}b^2)]^{-1} [1 - b^2R^{-2}]$

$\quad S_R = P_a \left[A+B[1-b^2R^{-2}] \left[A+B[1 - b^2a^{-2}]\right]^{-1}\right]$

c) $\rho = 1 + \left[\dfrac{b^{-2} - a^{-2}}{4}\right]^{-1} \left[\dfrac{P_b a^{-2} + P_a b^{-2}}{A+B} - \dfrac{P_b+P_a}{B} R^{-2}\right]$

$\quad S_R = [b^{-2} - a^{-2}]^{-1} \left[P_b[a^{-2} - R^{-2}] + P_a [b^{-2} - R^{-2}]\right]$

<u>Material Class II</u>

$$W = A(p^2+q^2) + B(p^2-q^2) - (2A+B) - 4(B+A)p \qquad 2.6.28$$

This has a kernel of the form $W = A(I-2)+B(J-1)$

a) $\rho = 1 + \dfrac{P_a}{4(B-A)} \left(\dfrac{a}{R}\right)^2$, $\quad S_R = P_a \left(\dfrac{a}{R}\right)^2$

b) $\rho = 1 + \dfrac{P_a[1-b^2a^{-2}]^{-1}}{4(B-A)} \left[1 - \dfrac{b^2}{R^2}\right]$

$$S_R = P_a[b^2R^{-2}-1][b^2a^{-2}-1]^{-1}$$

c) $$\rho = 1+\left[\frac{b^{-2}-a^{-2}}{4}\right]^{-1}\left[\frac{P_b a^{-2}+P_a b^{-2}}{4[B+A]} - \frac{[P_b+P_a]R^{-2}}{4[A-B]}\right]$$

$$S_R = [b^{-2}-a^{-2}]^{-1}\left[P_b[a^{-2}-R^{-2}] - P_a[R^{-2}-b^{-2}]\right]$$

Material Class III

$$W = B(p^2- q^2)+Lq^{\alpha+1}-4Bp-4B \quad \alpha\neq -1,0 \tag{2.6.29}$$

a) $$\rho = 1+P_a a^{2/\alpha} R^{-2/\alpha}\left[4B-\frac{2(\alpha+1)L}{\alpha}\right]^{-1}$$

$$S_R = P_a\left(\frac{a}{R}\right)^{2/\alpha}$$

b) $$\rho = 1+P_a\left[4B[a^{-2/\alpha}-b^{-2/\alpha}]-\frac{2(\alpha+1)La^{-2/\alpha}}{\alpha}\right]^{-1}[b^{2/\alpha}R^{-2/\alpha}-1]$$

$$S_R = P_a\left[4B[a^{-2/\alpha}-b^{-2/\alpha}]-\frac{2(\alpha+1)La^{-2/\alpha}}{\alpha}\right]^{-1}\left[4B[R^{-2/\alpha}-b^{-2/\alpha}]-\frac{2(\alpha+1)R^{-2/\alpha}}{\alpha}\right]$$

c) $$\rho = 1+[b^{-2/\alpha}-a^{-2/\alpha}]^{-1}\left[\frac{[P_b a^{-2/\alpha}+P_a b^{-2/\alpha}]}{4B}-\frac{[P_b-P_a]R^{-2/\alpha}}{\left[4B-\frac{2(\alpha+1)L}{\alpha}\right]}\right]$$

$$S_R = [a^{-2/\alpha}-b^{-2/\alpha}]^{-1}\left[P_b[R^{-2/\alpha}-a^{-2/\alpha}]+P_a[R^{-2/\alpha}-b^{-2/\alpha}]\right]$$

Material Class IV $\quad \sigma_{ij} = 2\mu e_{ij}+\lambda\delta_{ij}e_{kk}$ 2.6.30

The classical linear material

a) $$\rho = 1-\frac{P_a a^2}{3\lambda+2\mu}, \quad S_R = P_a(a/R)^2$$

b) $$\rho = 1+P_a[(3\lambda+\mu)a^{-2}+b^{-2}(\mu+2\lambda)]^{-1}[b^{-2}-R^{-2}]$$

$$t_R = P_a[(3\lambda+\mu)a^{-2}+b^{-2}(\mu+2\lambda)]^{-1}[R^{-2}(3\lambda+\mu)-b^{-2}(\mu+2\lambda)]$$

c) $$\rho = 1+[b^{-2}-a^{-2}]^{-1}\left[\frac{P_b a^{-2}+P_a b^{-2}}{(\mu+2\lambda)}+\frac{(P_a+P_B)R^{-2}}{(3\lambda+\mu)}\right]$$

$$t_R = (b^{-2}-a^{-2})^{-1}[P_b(a^{-2}-R^{-2})+P_a(b^{-2}-R^{-2})]$$

(See Appendix A1 for a summary of the manner in which these solutions for the classical material were obtained).

Inspection of the detailed solutions reveals that apart from material class III, the form is very much the same as that for the classical solution, corresponding to

$$\rho = C + DR^{-2}$$

and 2.6.31

$$S_R = Pl^{-2}$$

where C and D are constants, P some pressure term and l is some characteristic length of the problem. No further analysis of the solutions is presented. The solution form 2.6.31 will be reproduced in a later chapter after a new technique has been developed. The form 2.6.31 is seen to be associated with the fact that p is constant.

As for the material class III there is good reason to doubt its validity. As was indicated in Section 2.5, it is a reasonable requirement that a defined material class becomes asymptotically linear for small strains. The material of class III does not satisfy this requirement, unless $\alpha = 1$. The material class must, however, not be discounted as it may well be valid away from the origin in strain space. The equivalence for small strains may be employed to suitably restrict the parameters in the material classes I and II.

CHAPTER 3 A NEW APPROACH

SECTION 3.1 INTRODUCTION

In this chapter a new approach to the solution of the equilibrium equations for plane strain is introduced and developed. In developing the new approach an assumption is made concerning the form of the equilibrium equations. The adoption of this assumption has the effect of reducing the class of problems which may be considered to that consistent with this assumption. The restriction of consistency is discussed. The latter part of this chapter consists of example solutions illustrating the application of the new approach.

In Section 3.2 the formalism and notation is introduced. The equilibrium equations are assumed to decouple into a pair of Cauchy-Riemann equations. The consequences of this assumption are discussed and, in particular, the effect on the field variables is examined. Section 3 contains a discussion of the complementary formulation in terms of $Wc(p_T, q_T)$ The duality as considered in Section 2.4 is again surveyed. In Section 3.4 the general solution of the equations is generated for particular materials and the restrictions imposed by the assumption of Section 3.2 are incorporated. Two problems are solved for a particular material class and the impact of the assumption of Section 2 on another class is considered. No further problems are discussed as the approach of this chapter is eclipsed by another presented in Chapter 4, where closed form analytic solutions are obtained. Finally, in Section 3.5 the method is compared and contrasted with one which provided much motivation for that in this chapter, the method of JOHN (1960).

The work presented in this chapter has been published as "TOWARDS THE SOLUTION OF FINITE PLANE-STRAIN PROBLEMS FOR COMPRESSIBLE ELASTIC SOLIDS" in Int.J.Solids Structures, 1977, Vol.13, pp105-123, the paper was co-authored with DR. R. W. OGDEN.

SECTION 3.2 FORMALISM AND ITS CONSEQUENCES

Starting with equation 2.4.13(2), which may be expanded thus

$$\begin{aligned} S_{11} &= W_p \cos\chi + W_q \cos\chi^* \\ S_{12} &= W_p \sin\chi + W_q \sin\chi^* \\ S_{21} &= W_p \sin\chi + W_q \sin\chi^* \\ S_{22} &= W_p \cos\chi - W_q \cos\chi^*, \end{aligned} \qquad 3.2.1$$

and introducing the notation

$$A = W_p \cos\chi, \qquad B = W_q \sin\chi$$

with 3.2.2

$$C = W_q \cos\chi^*, \qquad D = W_q \sin\chi,$$

we may write 3.2.1 in the compact form

$$S_{11} = A + C, \qquad S_{12} = -B + D$$

$$S_{21} = B + D, \qquad S_{22} = A - C. \qquad 3.2.3$$

It follows from 3.2.2 that

$$W_p = (A^2 + B^2)^{\frac{1}{2}}$$

and 3.2.4

$$W_q = (C^2 + D^2)^{\frac{1}{2}},$$

and from 2.3.9, 2.3.4, 2.4.17 and 3.2.2 that

$$\operatorname{Tan}\chi = \frac{\alpha_{12}-\alpha_{21}}{\alpha_{22}+\alpha_{11}} = \frac{S_{21}-S_{12}}{S_{11}+S_{22}} = \frac{B}{A}, W_p \neq 0 \qquad 3.2.5$$

and

$$\operatorname{Tan}\chi^* = \frac{\alpha_{12}+\alpha_{21}}{\alpha_{11}-\alpha_{22}} = \frac{S_{12}+S_{21}}{S_{11}-S_{22}} = \frac{D}{C}, W_q \neq 0,\ q \neq 0. \qquad 3.2.6$$

From these equations, χ and χ^* may be determined to within an integral multiple of π, for a given $\underline{S}$. This is so provided 2.2.4(1) and (2) are not zero, in which case χ and χ^* become indeterminate, respectively. Using 2.3.10, 2.3.13, 2.4.4 and 2.4.14, equations 3.2.5 and 3.2.6 may be manipulated to yield

$$\alpha_{11}+\alpha_{22} = 4AWc_{p_T}/p_T$$

$$\alpha_{11}-\alpha_{22} = 4CWc_{p_T}/p_T$$

and 3.2.7

$$\alpha_{12}-\alpha_{21} = 4BWc_{q_T}/q_T$$
$$\alpha_{12}+\alpha_{21} = 4DWc_{q_T}/q_T ,$$

provided $p_T \neq 0$. The existence of Wc is discussed in Chapter 1.

Now, using the relationships

$$p_T = 2(A^2+B^2)^{\frac{1}{2}}, \quad q_T = 2(C^2+D^2)^{\frac{1}{2}} \qquad 3.2.8$$

which are obtained from 2.4.2 and 3.2.3, the right hand sides of equations 3.2.7 are demonstrably functions of A, B, C and D. When these are known, and providing Wc can be determined (See OGDEN (1977) or Chapter 1), the components of $\underline{\alpha}$ can be determined (from 3.2.7). They must satisfy the compatibility equations 2.2.3.

Introducing the notation

$$P_T = Wc_{p_T}/p_T, \quad Q_T = Wc_{q_T}/q_T \qquad 3.2.9$$

and using 3.2.7, the deformation gradient components may be written as

$$\begin{aligned} \alpha_{11} &= 2(P_TA+Q_TC), \ \alpha_{12} = 2(P_TB+Q_TD) \\ \alpha_{21} &= 2(-P_TB+Q_TD), \ \alpha_{22} = 2(P_TA-Q_TC). \end{aligned} \qquad 3.2.10$$

In summary, the basic problem specified by equations 2.2.1 and 2.2.2 may be replaced by four first order partial differential equations in A, B, C and D. Specifically,

$$\begin{aligned} (A+C)_{,1} + (B+D)_{,2} &= 0 \\ (-B+D)_{,1} + (A-C)_{,2} &= 0 \end{aligned} \qquad 3.2.11$$

with

$$\begin{aligned} (P_TA+Q_TC)_{,2} - (P_TB+Q_TD)_{,1} &= 0 \\ (-P_TB+Q_TD)_{,2} - (P_TA-Q_TC)_{,1} &= 0 . \end{aligned} \qquad 3.2.12$$

Considering boundary conditions, let the boundary of the plane region occupied by the material be denoted by Σ in the undeformed configuration. Assume also that the unit tangent to Σ be $(\mathrm{Cos}\psi, \mathrm{Sin}\psi)$, then the components (t_1,t_2) of the traction per unit (undeformed) area are given by

$$\begin{aligned} t_1 &= W_p\mathrm{Sin}(\psi-\chi) + W_q\mathrm{Sin}(\psi-\chi^*) \\ &= n_1(A+C) + n_2(B+D) \\ t_2 &= W_p\mathrm{Cos}(\psi-\chi) + W_q\mathrm{Cos}(\psi-\chi^*) \\ &= n_1(-B+D) + n_2(A-C), \end{aligned} \qquad 3.2.13$$

where (n_1,n_2) denotes the unit normal to Σ. Equations 3.2.13 are obtained by contraction of the nominal stress with a normal to the surface (Equations 1.3.3, 2.4.13(3) and 3.2.2). The functions A, B, C and D must be consistent with the traction boundary conditions in $\mathcal{E}$. The solution for x_1 and x_2 from 3.2.10 must satisfy the boundary conditions of place where specified in $\mathcal{E}$.

The equilibrium equations, 2.2.1, can be satisfied identically if stress functions h_1 and h_2 are introduced such that

$$S_{11} = h_{2,2},\ S_{21} = -h_{2,1},\ S_{12} = -h_{1,2} \text{ and } S_{22} = h_{1,1}. \qquad 3.2.14$$

Using these, equations 3.2.3 may be rewritten to yield

$$\begin{aligned} 2A &= h_{1,1}+h_{2,2} \quad & 2B &= h_{1,2}-h_{2,1} \\ 2C &= h_{2,2}-h_{1,1} \quad & 2D &= -h_{1,2}-h_{2,1} \end{aligned} \ . \qquad 3.2.15$$

The relationships

$$\begin{aligned} \nabla^2 h_1 &= 2(A_{,1}+B_{,2}) = -2(C_{,1}+D_{,2}) \\ \nabla^2 h_2 &= 2(A_{,2}-B_{,1}) = 2(C_{,2}+D_{,1}) \end{aligned} \qquad 3.2.16$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}} \quad ,$$

should also be noted.

It will be noted from 3.2.11 and 3.2.16 that harmonic $h_i, i = 1,2$ are sufficient conditions for the equilibrium equations to be satisfied. It is this assumption which will be adopted, viz. the harmonicity of h_1 and h_2. It must be emphasised that this assumption is but one of many that could be made. It is adopted here merely to illustrate a technique.

Up until now the analysis has been completely general, given the stated field of interest. W is as yet unrestricted save that Wc is assumed to exist in some neighbourhood of the natural configuration in strain space. The assumption of harmonicity of the stress potentials restricts W to those materials for which the problems considered have a solution consistent with this assumption. No assumptions have to be made and problems can be solved in complete generality. Indeed, the solutions presented at the end of Chapter 2 could be presented in the current formalism. This will be demonstrated in the next and subsequent chapters.

In the singular case, when $W_p = 0$ and the angle χ indeterminate, then the equation 2.2.9(1) is no longer valid. However, in this case A = B = 0 and, $2P_T A$ and $2P_T B$ must be replaced by $Wc_{p_T} \cos\chi$ and $Wc_{p_T} \sin\chi$ in 3.2.20, respectively, where χ is now taken as arbitrary. This, together with the boundary conditions serves to determine χ. Reducing 3.2.15 employing the fact that A = B = 0, the following is obtained

$$2C = \phi_{,22} - \phi_{,11}\ , \qquad D = -\phi_{,12}$$

with 3.2.17

$$\nabla^2\phi = 0.$$

The problem reduces to finding the harmonic function ϕ together with χ, consistent with the boundary data.

The consequences of the stress potentials being harmonic are many. Firstly, as was noted in Section 3.1, the equilibrium equations effectively decouple to become

$$\begin{aligned} A_{,1} + B_{,2} &= 0 \qquad -B_{,1} + A_{,2} = 0 \\ C_{,1} + D_{,2} &= 0 \qquad -D_{,1} + C_{,2} = 0. \end{aligned} \qquad 3.2.18$$

Thus (A,B) and (C,D) are conjugate harmonic pairs, and functions ϕ, ϕ^*, ψ and ψ^* may be introduced such that

$$\begin{aligned} A &= \phi_{,1} = \psi_{,2} \qquad B = \phi_{,2} = -\psi_{,1} \\ C &= \phi^*_{,1} = \psi^*_{,2} \qquad D = \phi^*_{,2} = -\psi^*_{,1} \end{aligned} \qquad 3.2.19$$

with

$$\nabla^2\phi = \nabla^2\phi^* = \nabla^2\psi = \nabla^2\psi^* = 0\ . \qquad 3.2.20$$

Equations 3.2.18 may also be employed to demonstrate that the compatibility equations 3.2.12 may be written as

$$\begin{aligned} \nabla P_T \cdot \nabla\phi \; - \nabla Q_T \cdot \nabla\phi^* &= 0 \\ \nabla P_T \wedge \nabla\phi \; + \nabla Q_T \wedge \nabla\phi^* &= 0, \end{aligned} \qquad 3.2.21$$

in the conventional dyadic notation (i.e. $\cdot$ scalar product, $\wedge$ vector product and ∇ the rector operator $(\partial/\partial X_1, \partial/X_2)$). In addition, using 3.2.5 with 3.2.18, it may be shown that

$$\nabla^2\chi = \nabla^2\chi^* = 0\ . \qquad 3.2.22$$

When used with the equilibrium equations, these equations also serve to demonstrate that

$$\nabla^2 W_p = W_{p,i}\chi_{,i} = \nabla^2 W_q = W_{q,i}\chi^*_{,i} = 0 \quad . \qquad 3.2.23$$

The consequence of paramount importance, however, is that for a given problem only a restricted class of materials is suitable for consideration. The quantities P_T and Q_T in equation 3.2.21 are both functions of $\nabla\phi$ and $\nabla\phi^*$ through equations 3.2.8, 3.2.9 and 3.2.19. In general, the conditions represented by 3.2.21 are incompatible with 3.2.22. This does, of course, reflect the fact that an assumption has been injected into a self-consistent, closed system. However, in certain circumstances 3.2.21 and 3.2.22 are consistent. It is this that defines the soluable material/problem class, given the assumption of harmonic h_1 and h_2. A trivial case in which this is true is when p and q are constant, and the deformation homogeneous; a state achievable by all materials given suitable boundary data. Another case, as will be seen subsequently in Section 3.4 is that of harmonic materials when radially symmetric deformations are considered (see equation 2.6.27, material class I).

The components of traction (nominal) $t_i, i=1,2$ on Σ can be written

$$\begin{aligned} t_1 &= \underline{M} \cdot \nabla(\psi+\psi^*) \\ t_2 &= \underline{M} \cdot \nabla(\phi^*-\phi) \end{aligned} \qquad 3.2.24$$

where $\underline{M}$ is the unit tangent to Σ measured in the positive sense. These relationships are generated as for 3.2.13. $\psi+\psi^*$ and $\phi^*-\phi$ are simply the stress potentials h_1 and h_2 respectively.

Thus summarising; by adopting a material class and problem consistent with harmonicity of the stress potentials, the solution is derived as follows:-

i) Determine h_i harmonic inside Σ such that 3.2.24 holds where tractions are specified.

ii) Evaluate the function A, B, C and D from 3.2.15.

iii) Integrate 3.2.19 to yield ϕ, ϕ^*.

iv) Incorporate ϕ, ϕ^* into 3.2.21.

v) Again employ 3.2.19, this time to obtain A, B, C and D, which are likely to be simplified from their form at stage ii.

vi) Use 2.2.9 to find P_T and Q_T.

vii) 2.2.10 may now be used to obtain $\underline{\alpha}$

viii) Integrate the $\underline{\alpha}$ obtained at stage vi to obtain x_i, i=1,2 (Stage vi ensures that this integration is possible).

ix) Incorporate boundary conditions of place where specified.

The advantage of this formulation over that conventionally adopted is that a schema can be defined for the solution where each stage is given in terms of well studied problems. Harmonicity has the additional advantage that complex variable theory may be invoked. The schema identified above is illustrated in Section 3.4.

SECTION 3.3 THE DUAL FORMULATION

An alternative approach to that described in Section 3.2 will begin with the strain compatibility equations 2.4.11(a).

Now,

$$\begin{aligned}
\alpha_{11} &= Wc_{p_T}\,\mathrm{Cos}\,\chi + Wc_{q_T}\,\mathrm{Cos}\,\chi^* \\
\alpha_{12} &= -Wc_{p_T}\,\mathrm{Sin}\,\chi + Wc_{q_T}\,\mathrm{Sin}\,\chi^* \\
\alpha_{21} &= Wc_{p_T}\,\mathrm{Sin}\,\chi + Wc_{q_T}\,\mathrm{Sin}\,\chi^* \\
\alpha_{22} &= Wc_{p_T}\,\mathrm{Cos}\,\chi - Wc_{q_T}\,\mathrm{Cos}\,\chi \quad .
\end{aligned} \qquad 3.3.1$$

Continuing as in Section 3.2 and introducing

$$\begin{aligned}
\bar{A} &= Wc_{p_T}\,\mathrm{Cos}\,\chi & \bar{B} &= Wc_{p_T}\,\mathrm{Sin}\,\chi \\
\bar{C} &= Wc_{q_T}\,\mathrm{Cos}\,\chi^* & \bar{D} &= Wc_{q_T}\,\mathrm{Sin}\,\chi
\end{aligned} \qquad 3.3.2$$

and rewriting 3.3.1 as

$$\begin{aligned}
\alpha_{11} &= \bar{A} + \bar{C} & \alpha_{12} &= \bar{B} + \bar{D} \\
\alpha_{21} &= -\bar{B} + \bar{D} & \alpha_{22} &= \bar{A} - \bar{C}
\end{aligned} \qquad 3.3.3$$

then the relationships

$$\tfrac{1}{2}p = Wc_{p_T} = (\bar{A}^2 + \bar{B}^2)^{\frac{1}{2}}$$

and $\qquad 3.3.4$

$$\tfrac{1}{2}q = Wc_{q_T} = (\bar{C}^2 + \bar{D}^2)^{\frac{1}{2}}$$

are easily obtained from 3.3.2 and 2.4.14.

Now, adopting $x_i, i = 1,2$ as the primitive fields, the compatibility equations are satisfied identically. Equation 3.3.3 is employed to yield

$$\begin{aligned}
2\bar{A} &= x_{1,1} + x_{2,2} & 2\bar{B} &= x_{1,2} - x_{2,1} \\
2\bar{C} &= x_{1,1} - x_{2,2} & 2\bar{D} &= x_{1,2} + x_{2,1} \quad .
\end{aligned} \qquad 3.3.5$$

Then using 3.3.2 and 2.4.13 with P and Q defined as

$$P = W_p/p, \qquad Q = W_q/q, \qquad 3.3.6$$

the stress components may be written as

$$\begin{aligned}
S_{11} &= 2(P\bar{A} + Q\bar{C}), & S_{12} &= 2(-P\bar{B} + Q\bar{D}) \\
S_{21} &= 2(P\bar{B} + Q\bar{D}), & S_{22} &= 2(P\bar{A} - Q\bar{C}).
\end{aligned} \qquad 3.3.7$$

The equilibrium equations become

$$(P\overline{A} + Q\overline{C})_{,1} + (P\overline{B} + Q\overline{D})_{,2} = 0$$
$$(-P\overline{B} + Q\overline{D})_{,1} + (P\overline{A} - Q\overline{C})_{,2} = 0 \qquad 3.3.8$$

These last two equations, being of a similar form, are comparable with 3.2.12. The results

$$2(\overline{A}_{,1} + \overline{B}_{,2}) = 2(\overline{C}_{,1} + \overline{D}_{,2}) = \nabla^2 x_1$$

and

$$2(\overline{A}_{,2} - \overline{B}_{,1}) = -2(\overline{C}_{,2} - \overline{D}_{,1}) = \nabla^2 x_2$$

are also obtainable and are analogous to 3.2.6.

As an alternative to solving 3.3.8 directly (which is unlikely to be a trivial task), assumptions regarding the x_i could be made in a similar manner to those about h_i in Section 3.2. This avenue is not developed and only the method of Section 3.2 is illustrated. The duality of formulation has only been considered in order to expose an elegant symmetry underlying the general theory, at least in plane strain.

SECTION 3.4 SOLUTION OF PROBLEMS

Prior to solving any problems, the restrictions placed on two material classes, given the assumption that the stress potentials h_i are harmonic, will be investigated.

CLASS I

The class of harmonic materials (2.3.17 - 18, JOHN [1960]) is written as

$$W = f(p) + \tfrac{1}{2}\mu q^2 , \qquad 3.4.1$$

where μ is the conventional shear modulus and f(p) some function.

Now, from 3.2.9 and 2.4.14,

$$Q_T = \frac{Wc_{q_T}}{q_T} = \frac{1}{4}\frac{q}{W_q} , \qquad 3.4.2$$

and thus, for the form 3.4.1,

$$Q_T = \tfrac{1}{4}\mu \text{ a constant,} \qquad 3.4.3$$

and in particular

$$\nabla Q_T = 0. \qquad 3.4.4$$

The compatibility equations as written in 3.2.21, become

$$\nabla P_T = 0 \qquad 3.4.5$$

provided $\nabla\phi \neq 0$. If $\nabla\phi$ were zero, A and B would also be zero and χ would be indeterminate. The degenerate case leading to equation 3.2.17 would result. It follows from 3.4.5 that either

$$\text{(i)} \quad f(p) \propto p^2$$

$$\text{or} \qquad 3.4.6$$

$$\text{(ii)} \quad p \text{ constant}$$

(See equations 2.6.21, which were generated generally for this material class but for a restricted deformation class).

The solution (i) is untenable in that the corresponding energy density function would not represent a realistic material. In particular, such a material would not be stress-free at zero strain, when p = 2. To be realistic, the results of Section 2.5 must be applied, in which case, for small strains, the form 3.4.6 must be such that

$$f(2)=0, \quad f'(2)=0, \quad f''(2)=\lambda+\mu, \qquad 3.4.7$$

contradicting the adoption of solution (i) above. The semi-linear material of 2.5.19 is the minimal strain energy function satisfying these constraints.

That p is constant is a necessary consequence of the assumption of harmonic stress potentials h_i, for this class of materials. This does not mean that p is forever constant for all materials of Class I. It simply means that the technique as applied is consistent and valid only for problems where p happens to be constant for materials of Class I. Other restrictions would naturally lead to other problems being solvable by analogous techniques.

CLASS II

This class is the class of materials with strain energy density functions of the form

$$W = \tfrac{1}{2}\mu p^2 + g(q). \qquad 3.4.8$$

The result

$$\nabla P_T = 0 \qquad 3.4.9$$

is derived as in 3.4.4, from 3.2.9 and 2.4.14. This may be employed to simplify the compatibility equations 3.2.21, to yield

$$\nabla Q_T = 0 \qquad 3.4.10$$

and hence,

$$q \text{ constant} \qquad 3.4.11$$

provided $g(q) \neq Kq^2$, some K.

However, this class of material is only of marginal interest in that the undeformed state is only maintainable by the application of a hydrostatic stress magnitude 2μ. Indeed, this class of materials could be precluded from consideration on the basis of the argument used to discount 3.4.6(i).

In considering the two classes of material it is seen that solutions to problems in which either p or q are constant may be generated. Section 2.6 contains solutions of the form p is constant, for harmonic materials and as such the technique as developed may be employed. In employing this technique further insight as to the forms of W consistent with harmonic h_1 and h_2 is gained.

In the case of radial symmetry, equation 2.6.3 demonstrates that the deformation gradient $\underset{\sim}{\alpha}$ is symmetric. Then equations 2.3.9 and 3.2.5 imply

$$\chi = 0 \text{ and } B = 0, \qquad 3.4.12$$

respectively. Hence, from 3.2.18(1) it can be seen that

$$A \text{ is constant.} \qquad 3.4.13$$

Thus, from 2.4.14 and 3.2.19,

$$\tfrac{1}{2}p_T = W_p\ (p,q) = A \text{ is constant}$$

and $\qquad 3.4.14$

$$\tfrac{1}{2}q_T = W_q\ (p,q) = (C^2+D^2)^{\frac{1}{2}} = |\nabla\phi^*|$$

with $\nabla^2\phi^* = 0$. Additionally, when equation 3.3.16 is used

$$\chi^* = 2\theta_E \qquad 3.4.15$$

is obtained, where θ_E is the angle of orientation of the Eulerian axes. To obtain the result

$$D = C\tan 2\theta_E \qquad 3.4.16$$

equation 3.2.6 may be employed.

Now, as indicated above, $\underset{\sim}{\alpha}$ is symmetric for this class of problem. Consequently, when the polar decomposition theorem of 1.2.3 is invoked, the rotation is found to be the identity transformation. Hence the angle θ_E may be identified with the conventional polar angle, θ, there being no local rotation. Equations 3.4.14 thus yields

$$\tfrac{1}{2}q_T = C\sec 2\theta = \frac{CR^2}{X_1^{\,2}-X_2^{\,2}}$$

and 2.6.23 may be generalised to obtain an expression for S_R, the nominal radial pressure

$$S_R = W_p + W_q \quad . \qquad 3.4.17$$

Then, imposing 3.4.13 and invoking the essential symmetry of the problem class, in that S_R is a function of R only, it can be concluded that

$$W_q = \tfrac{1}{2}q_T \qquad 3.4.18$$

is a function of R only.

In proceeding, the system

$$\nabla^2\phi^* = 0$$

with 3.4.19

$$\frac{\partial |\nabla \phi^*|}{\partial \theta} = 0$$

must be solved, in order that C and D may be found.

Now, as ϕ^* is harmonic, it may be represented thus

$$\phi^* = \mathrm{Re}(F(z)), \qquad 3.4.20$$

for some analytic $F(z)$ where $z = X_1 + iX_2$. The region of analycity is such that z is some material point of interest. The Cauchy-Riemann equations then yield

$$\phi^*{,}_1 - i\phi^*{,}_2 = F(z)_{,z} \ . \qquad 3.4.21$$

Consequently

$$|\nabla \phi^*| = |F(z)_{,z}|$$
$$= (F(z)_{,z}\overline{F(z)_{,z}})^{\frac{1}{2}} \ . \qquad 3.4.22$$

Now, if $F(z)$ is represented as a power series in z, the application of 3.4.22 results in a doubly infinite set of equations (non-linear) to be solved for an infinite set of coefficients.

However, consider

$$F(z) = \frac{1}{e+1} \ a_{e+1} \ z^{e+1} \qquad e \neq -1. \qquad 3.4.23$$

This is recognised to be a general term of a power series if e were to be an integer, but e is not so restricted. Then

$$|\nabla \phi^*| = |a_{e+1}| R^{e} \qquad 3.4.24(1)$$

and is indeed independent of θ. It would not be surprising if this solution set were unique, the reason being that the governing equations for finite elastostatics are demonstrably well posed. In addition, the system is second order and suitable restrictions on W guarantee that the system is elliptical. The consequence of this is that at least in plane strain, two boundary conditions (or limits) have to be satisfied. Indeed, in the problem class in question, radial symmetry, these conditions are to be applied at points only. The problem posed is fully solved once $q_T = 2|\nabla \phi^*|$ has been determined. Too many undetermined parameters cannot be admitted as there are only two boundary conditions to be satisfied. The solutions presented in Section 2.6 correspond to $e = -2$ for all materials apart from

Letting

$$/a_{e+1}/ = \alpha > 0, \qquad 3.4.24(2)$$

it may be concluded that

$$q_T = 2\alpha R^e$$

and 3.4.25

$$C = \alpha R^e \mathrm{Cos}2\theta \;, \quad D = \alpha R^e \mathrm{Sin}2\theta \;,$$

where 3.3.2(2) and 3.4.18 have been used to generate equation 3.4.25.

Then, using 3.4.12, 3.4.13 and 3.4.25 to substitute into 3.2.3, the nominal stress field is obtained as follow:

$$S_{11} = A + \alpha R^e \mathrm{Cos}2\theta$$

$$S_{12} = S_{21} = \alpha R^e \mathrm{Sin}2\theta \qquad 3.4.26$$

$$S_{22} = A - \alpha R^e \mathrm{Cos}2\theta.$$

Thus it may be noted that the method (harmonic h_i implying the existence of $\emptyset^*$) is consistent with materials which demonstrate power decay (growth) of nominal stress.

Now, remembering the definitions 3.3.29 for P_T and Q_T, and applying 3.4.14,

$$P_T = p/4A$$

and 3.4.27

$$Q_T = q/4\alpha R^e,$$

the strain compatibility equations (3.2.21) become

$$p' - q' = 2q/R, \qquad 3.4.28$$

where ()' denotes d/dR. This equation should be compared with those in Section 2.6, particularly 2.6.12(1). This form is independent of the harmonicity assumption, and in particular, of the value of e in 3.4.25. The admissible materials defined by W are restricted insofar as p and q, as determined from

$$W_p(p,q) = A$$

and 3.4.29

$$W_q(p,q) = \alpha R^e,$$

must be consistent with 3.4.28. No attempt will be made here to identify those classes.

Now, for materials of Class I

$$W_p \equiv f'(p) = A$$

and $\qquad$ 3.4.30

$$W_q = \mu q.$$

Thus imposing 3.4.29,

$$p = (f')^{-1}(A)$$
$$q = \alpha R^{e}/\mu \quad , \qquad 3.4.31$$

where $(f')^{-1}$ denotes the inverse of f', which is assumed to exist. Then, consistency on applying 3.4.28 requires that

$$e = -2,$$

and hence with this condition, consistency of the approach-material-problem set is achieved. The material Class III of Section 2.6 appears to be consistent for all e.

In terms of the variable ρ introduced in Section 2.6, the deformation is specified, via

$$\rho = \{1+\tfrac{1}{2}(f')^{-1}(A)\} R - \frac{\alpha}{2\mu R} \quad . \qquad 3.4.32$$

As 3.4.32 indicates for a solid cylinder, when R = 0 is allowed, the deformation is necessarily homogeneous. This is a result analogous to that obtained from the classical theory.

For the Class II materials (3.4.8), 3.4.29 becomes

$$p = A/\mu$$

and $\qquad$ 3.4.33

$$g'(q) = \alpha R^{e} .$$

Employing these in 3.4.28 yields an indeterminate result. Consistency depends intimately on g(q). If

$$g(q) \equiv Lq^{\gamma}$$

is adopted, then 3.4.28 yields

$$e = 2(1-\gamma)$$

as a requirement for consistency.

The discussion of this method is not continued, because a more general method will be demonstrated in Chapters 4 and 5. However, prior to leaving it entirely, the application to the problem of an annulus with an applied shear will be briefly outlined. The shear is assumed uniform as a function of Θ.

From 3.2.13, introducing the complex normal to Σ, n_1+in_2 the following expression for the complex nominal traction is obtained

$$S_1+iS_2 = (A-iB)(n_1+in_2) + (C+iD)(n_1-in_2) \quad . \qquad 3.4.34$$

Attention will be restricted to material Class I for the remainder of this chapter; substituting into 3.2.10

$$\begin{aligned} \alpha_{11} &= 2P_T A+C/2\mu, & \alpha_{12} &= 2P_T B+D/2\mu \\ \alpha_{21} &= 2P_T B+D/2\mu, & \alpha_{22} &= 2P_T A-C/2\mu \end{aligned} \qquad 3.4.35$$

is obtained. This may be integrated to determine the deformation field x_i. The earlier discussion of harmonic materials has demonstrated that p and hence P_T are constant. As a consequence of 3.2.18(1) it may be noted that A + iB is an analytic function of $\mathcal{Z} = X_1+iX_2$. The fact that P_T is constant implies that $|A + iB|$ is constant. A corollary of the MAXIMUM MODULUS theorem states that an analytic function having constant modulus is itself constant, hence both A and B are constant. Employing the potentials ϕ^* and ψ^* introduced in 3.2.19, the integral of 3.4.35 can be expressed as

$$\bar{x} = 2P_T(A+iB)\bar{\mathcal{Z}} + \tfrac{1}{2}\omega(\mathcal{Z})/\mu, \qquad 3.4.36$$

where $x = x_1+ix_2$, and where a constant term has been dropped as it corresponds to a rigid translation.

$$\omega(\mathcal{Z}) = \phi^*-i\psi^*$$

is an analytic function of $\mathcal{Z} = X_1+2X_2 = Re^{i\underline{\Phi}}$. It is noted that

$$C-iD = d\omega/\partial\mathcal{Z}. \qquad 3.4.37$$

The components x_i are single valued and as a consequence of 3.4.36, the same may be said of $\omega(\mathcal{Z})$, in which case

$$\omega(\mathcal{Z}) = \sum_{-\infty}^{\infty} a_n Z^n \qquad 3.4.38$$

is a valid (Laurent) expansion of $\omega(\mathcal{Z})$.

Consider now an elastic solid contained in an annular region radii $a(<1)$ and 1 in the undeformed state. The boundary data considered is as follows

$$x = z \text{ on } R = a$$

and 3.4.39

$$S_R + iS_{\bar{\Phi}} = iS \quad \text{on } R = 1$$

where S_R and $S_{\bar{\Phi}}$ are the radial and hoop components of the nominal stress respectively. They are given by

$$S_R + iS_{\bar{\Phi}} = A - iB + (C + iD)(n_1 - in_2)^2 \qquad 3.4.40$$

with

$$n_1 + in_2 = e^{i\bar{\Phi}} \quad . \qquad 3.4.41$$

Substitution of 3.4.36, 3.4.37 and 3.4.38 into 3.4.39, and employing 3.4.40 with 3.4.41 yields

$$\omega = a_{-1} z^{-1} \ , \qquad \text{(a)}$$

$$C - iD = a_{-1} z^{-2} \qquad \text{(b)}$$

and 3.4.42

$$x = 2P_T(A-iB)z + \frac{\overline{\bar{a}_{-1} z^{-1}}}{2\mu} \qquad \text{(c)}$$

with

$$a_{-1} = 2\mu a^2 \left(1-2P_T(A+iB)\right)$$

$$= \left(A + i(B+S)\right) , \qquad 3.4.43$$

the latter being sufficient to determine A and B in terms of S, once the form $f(p)$ has been selected. Using

$$P_T = \frac{W_{c\,p_T}}{p_T} = \frac{p}{4p_T} = (f')^{-1}\left((A^2+B^2)^{\frac{1}{2}}\right)/4(A^2+B^2)^{\frac{1}{2}}, \qquad 3.4.44$$

the deformation may be written as

$$r^2 = \left(\frac{a^2}{R^2} + 2P_T A\left(R-\frac{a^2}{R}\right)\right)^2 + \left(R-\frac{a^2}{R^2}\right) 4\, P_T{}^2 B^2 \qquad 3.4.45$$

and

$$\operatorname{Tan}(\theta-\bar{\Phi}) = \frac{-2P_T B(R-a^2/R^2)}{(a^2/R+2P_T A(R-a^2/R^2))} . \qquad 3.4.46$$

This form of the solution is greatly simplified if an additional radial traction magnitude A is applied on the surface R = 1. The solution becomes

$$r^2 = R^2 + \frac{\alpha}{4\mu^2}\left(\frac{1}{R} - \frac{R}{a^2}\right)^2 \qquad 3.4.47$$

where $a_{-1} = i\alpha$, $\alpha = (S+B) = -4\mu P_T B a^2$ and $2P_T A = 1$, with

$$\operatorname{Tan}(\theta - \Phi) = \frac{\alpha}{2\mu}\left(\frac{1}{a^2} - \frac{1}{R^2}\right). \qquad 3.4.48$$

It is worthy of note that with this radial traction applied, an automatic consequence is an overall volume increase. Equation 3.4.47 indicates that

$$r \geqslant R\ , \qquad 3.4.49$$

the equality being when R = a, where this is specified. In general the question of volume changes depends intimately on the material form through f(p). Also, it may be noted that $\operatorname{Tan}(\theta-\Phi)$ is a monotonic function of R, increasing from 0 at R = a to a maximum at R = 1. This is as expected.

For comparison, the corresponding classical results may be noted:

$$u_r = 0, \quad u_\theta = \frac{S^2}{2\mu}\left(\frac{r}{a^2} - \frac{1}{r}\right), \qquad 3.4.50$$

where (u_r, u_θ) is the polar displacement. The Cauchy stress (1.5.4) components, again polar, are

$$\sigma_{rr} = \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = \frac{S}{r^2} \qquad 3.4.51$$

From 3.4.50(i) it can be seen that the classical linear theory predicts an isochoric or volume preserving deformation. It is a significant point of interest that if this restriction were imposed under the assumption of harmonic h_i, then the resulting deformation is forced to be homogeneous, the boundary conditions of place force it to be an identity. Thus a volume preserving solution is not compatible with this material for this problem.

In order to illustrate the divergence of the solution (as in equations 3.4.45 and 3.4.46) from the classical results, the following form f(p) is chosen

$$f(p) = \tfrac{1}{2}(\lambda + \mu)(p-2)^2 + \frac{1}{24}\nu(p-2)^4 \qquad 3.4.52$$

This is seen to be an extension of the semi-linear introduced in 2.5.19. In order for this material to be "reasonable" in the sense of Section 2.5, the conditions presented there are appended as the condition

$$\nu > 0. \qquad 3.4.53$$

The material will then be reasonable in the required sense. However, further restrictions could feasibly be required to ensure a physically reasonable response for other problem classes. Indeed, f(p), of 3.4.52 will be further restricted, even for the problem considered here.

The following non-dimensional quantities are introduced here:-

$$\lambda^* = \lambda/\mu, \ \nu^* = \nu/\mu, \ f^* = f/\mu, \ A^* = A/\mu, \ B^* = B/\mu$$
$$\gamma^* = (A^{*2} + B^{*2})^{\frac{1}{2}}, \ P^* = \mu P_T, \ S^* = S/\mu \qquad 3.4.54$$

Then, given 3.4.52, 3.4.43 may be employed to obtain A and B, or equivalently A^* and B^*, in terms of the non-dimensional quantities

$$A^* = 2a^2(1+4P^*a^2)^{-1}$$
$$B^* = -S^2(1+4P^*a^2)^{-1} \qquad 3.4.55$$

where

$$P^* = \tfrac{1}{4}\gamma^{*-1}\left(\left(\frac{S^2}{a^4} + 4\right)^{\frac{1}{2}} - \frac{\gamma^*}{a^2}\right). \qquad 3.4.56$$

The quantity γ^* is determinable from

$$\gamma^* = f^{*\prime}(p)$$

and $\qquad 3.4.57$

$$p = \left(\frac{S^{*2}}{a^4} + 4\right)^{\frac{1}{2}} - \frac{\gamma^*}{a^2} \quad .$$

In non-dimensional form, the solution 3.4.45 with 3.4.46 becomes

$$\frac{r^2}{R^2} = \left(\frac{a^2}{R^2} + 2P^*A^*\left(1-\frac{a^2}{R^2}\right)\right)^2 + 4\left(1-\frac{a^2}{R^2}\right)^2 P^{*2}A^{*2} \qquad 3.4.58$$

and

$$\mathrm{Tan}(\theta-\Phi) = -2P^*B^*\left(\frac{R}{a}-\frac{a}{R}\right)/\left(\frac{a}{R}+2P^*A^*\left(\frac{R}{a}-\frac{a}{R}\right)\right) \qquad 3.4.59$$

respectively.

Calculations have been carried out for $a = \frac{1}{2}$ and $\frac{1}{4}$, and for a range of S^* from 0 to 1.5 in steps of 0.1. Values of $\lambda^* = 0.1$ and $\nu^* = 24$ have been assumed for material constants. The main features of the solution are insensitive to the values used.

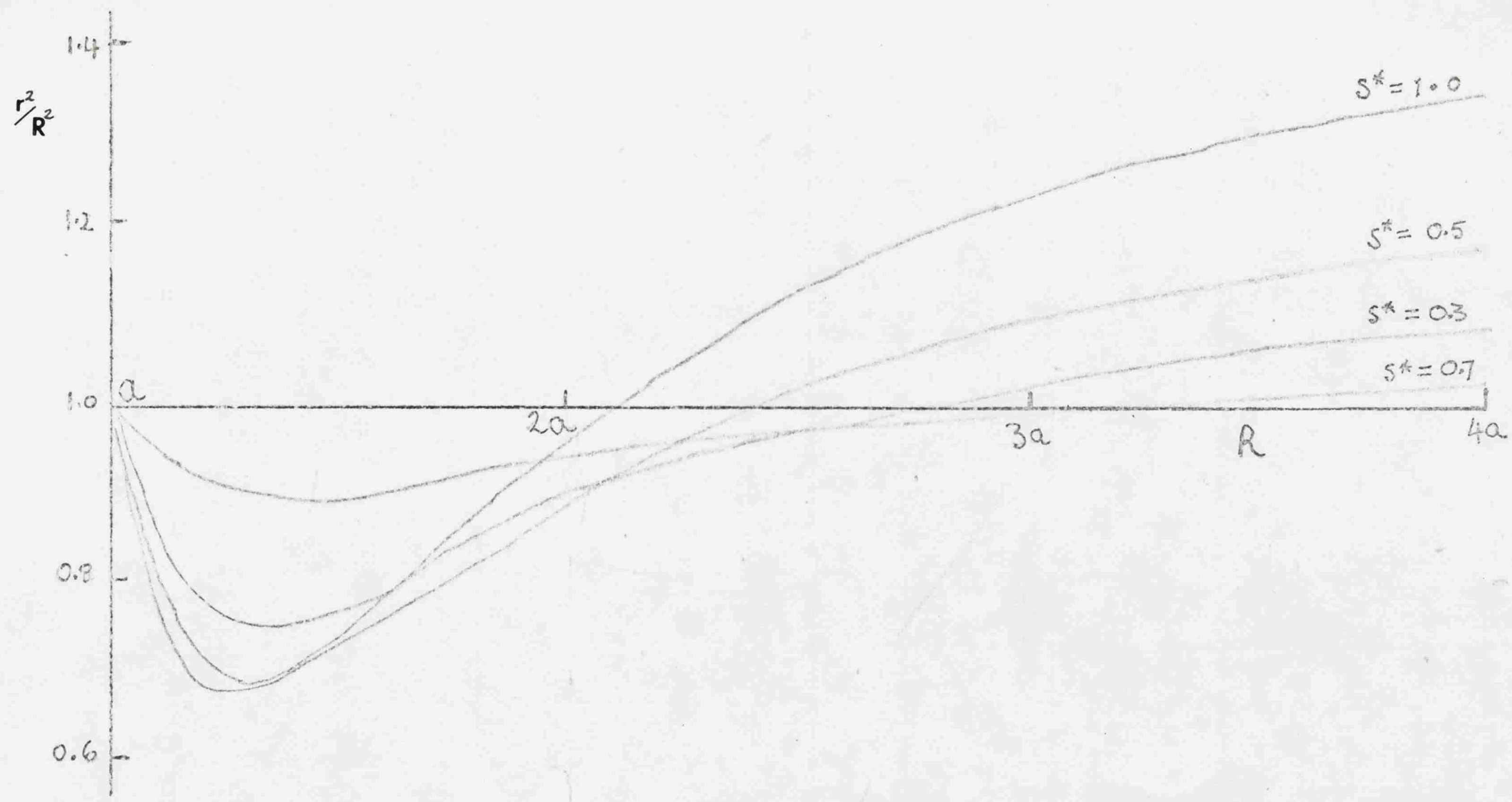


Fig 3.1

Figure 3.1 is a graphical representation of one set of results for equation 3.4.58. The case illustrated is for $a = \frac{1}{4}$ and $S^* = 0.1, 0.3, 0.5$ and 1.0. The measure r^2/R^2 represents the change of planar area, to be interpreted as volume in the plane strain case, inside a circle radius R. Notice, in particular, that near $R = a(=\frac{1}{4})$ there is a marked volume decrease. Overall, however, there is a volume increase for all cases, for this material at least.

The fact that a local decrease in volume near R = a is predicted is independent of the form of f(p). This is true because 3.4.58, 3.4.43 and 3.4.44 indicate that A, B and P_T are independent of R. Then

$$\frac{d}{dR}\left(\frac{r^2}{R^2}\right) = \frac{-2A^*}{a^3} < 0 \quad . \qquad 3.4.60$$

It is possible to draw the inequality conclusion of 3.4.60 by inspecting 3.4.55, and noting that P* is positive, and hence that A* is also positive. Thus as $r^2/R^2 = 1$ at R = a, the result 3.4.60 indicates a local volume reduction near R = a for all materials.

Equation numbers 3.4.61 — 62 have been omitted.

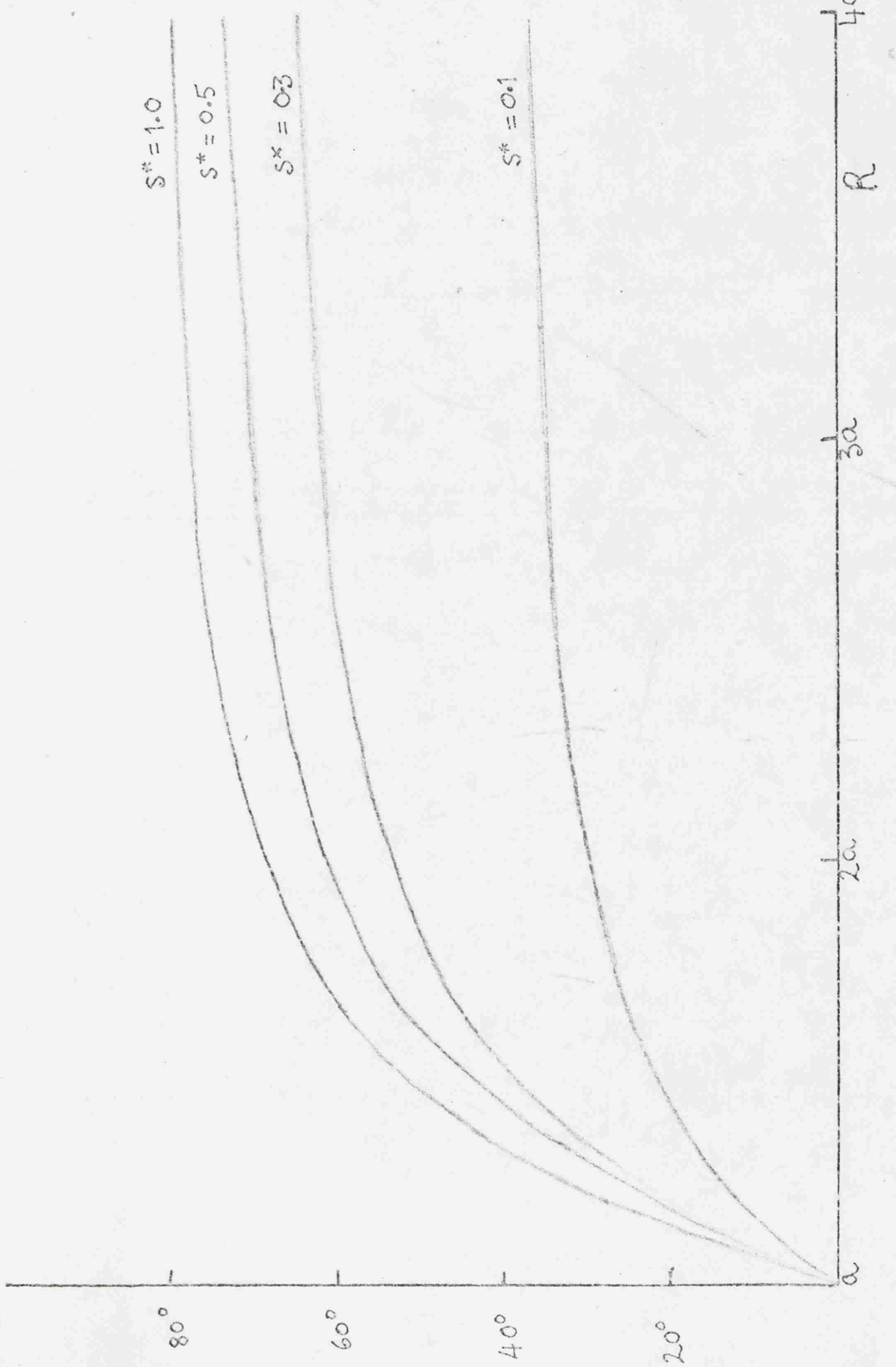


Fig 3.2

In Figure 3.2 curves of Θ-$\overline{\Phi}$ are plotted for S* = 0.1, 0.3, 0.5 and 1.0. They illustrate that shearing is greatest at R = a, decreasing monotonically with increasing R. This behaviour is to be expected since the boundary R = a is fixed. The association of an increased shear with reduction in 'a' is also to be anticipated.

Now, using 3.4.40, 3.4.42 and 3.4.44, the nominal traction on a surface normal $e^{i\overline{\Theta}}$ may be determined

$$S_R^* \equiv \mu^{-1} S_R = A^*(1-R^{-2}) \leq 0 \qquad 3.4.63$$

and

$$S_{\overline{\Phi}}^* \equiv \mu^{-1} S_{\overline{\Phi}} \doteq -B^* - \frac{4P^*B^*}{R} > 0 \qquad 3.4.64$$

when $S^* > 0$. It is of interest to obtain an estimate of the magnitude of the stresses at R = a. For the case $a = \frac{1}{4}$ and S* taken as 0.1

$$S_R^* \simeq -0.37$$

and **3.4.65**

$$S_{\overline{\Phi}}^* \simeq 1.2$$

is obtained. It is noteworthy that the shear stress has increased 12 fold over that applied.

The value of the associated principal stretches may also be estimated. Taking $\lambda_1 \geqslant \lambda_2$ then

$$\lambda_1 = \tfrac{1}{2}\,(p+q)$$

and

$$\lambda_2 = \tfrac{1}{2}\,(p-q)$$

where **3.4.66**

$$p = (f^{*\prime})^{-1}(\gamma^*)$$

and

$$q = 2((1-2P^*A^*)^2 + 4P^{*2}B^{*2})^{\frac{1}{2}} \; \frac{a^2}{R^2} \, .$$

For S* = 0.1 it is found that λ_1 has a maximum of 1.7 at R = a, decreasing monotonically to 1.06 at R = 1; whereas λ_2 increases monotonically from 0.36 to 0.98 over the same range. For higher values of S* the derived λ_2 can be negative and zero near R = a. This is physically unrealistic. This is a consequence of the chosen f(p).

The inequality

$$f'(p) < \mu p$$

is the necessary condition that must be satisfied in order that every $\lambda_1 > 0$ there is an associated $\lambda_2 > 1$ in pure shear. This condition was noted in JOHN (1960).

Finally, in this section a deficiency in the class of material considered, that of 3.4.1, is identified. As solved it can be seen that there is a limit as $S^* \to \infty$ when $\Theta - \bar{\Theta} \to \pi/2$. This in itself is unrealistic. However, it is not clear whether it is the material or method which has introduced this restriction. In any case the range of validity is sufficiently large for the discussion presented above to be reasonable.

Generally speaking the technique developed in this chapter is not of sufficiently general applicability to be developed further. Realistically, it is only of worth if, a'priori, some general properties of the solution are known, allowing the correct and consistent assumption to be made. In this case a semi-inverse method is likely to be of greater benefit. Further work that could be profitably undertaken is the analysis of composites consisting of annular rings of different materials. The Lagrangian formulation facilitates the fitting of the continuity boundary conditions. It would be of interest to investigate the volume change and the influence that the internal boundaries have on the stress concentration.

SECTION 3.5 COMPARISON WITH JOHN'S FORMULATION

In JOHN (1960) plane strain problems for materials with a strain energy density function of the form 3.4.6 are considered. The exact form of the strain energy density considered there, is

$$W = F(p) - 2\mu J, \qquad 3.5.1$$

where $J = \det[\alpha]$ is as introduced in 1.2.2.

Using 2.3.4 (2.3.5), 2.3.11, 2.3.12 and 2.3.19, the nominal stress components $\underline{S}$ may be written as

$$\begin{aligned} S_{11} &= F'(p)\cos\chi - 2\mu\alpha_{22} \\ S_{21} &= F'(p)\sin\chi + 2\mu\alpha_{21} \\ S_{12} &= -F'(p)\sin\chi + 2\mu\alpha_{12} \\ S_{22} &= F'(p)\cos\chi - 2\mu\alpha_{11}. \end{aligned} \qquad 3.5.2$$

Then writing

$$A = F'(p)\cos\chi , \quad B = F'(p)\sin\chi \qquad 3.5.3$$

(the A and B here differ from those previously defined), the equilibrium equations reduce to

$$\begin{aligned} A,_1 + B,_2 &= 0 \\ A,_2 - B,_1 &= 0 , \end{aligned} \qquad 3.5.4$$

if it is assumed that the α_{ij} satisfy the compatibility equations 2.4.11(a). The functions A and B are thus seen to be conjugate harmonics.

The components of the deformation field x_i, are then found from 3.5.3, expressed with the aid of 2.3.12 and 2.3.11(1) in the form

$$\begin{aligned} x_{1,1} + x_{2,2} &= pA/(A^2+B^2)^{\frac{1}{2}} \\ x_{1,2} - x_{2,1} &= pB/(A^2+B^2)^{\frac{1}{2}} \end{aligned} \qquad 3.5.5$$

where

$$p = (F')^{-1} [(A^2+B^2)^{\frac{1}{2}}] . \qquad 3.5.6$$

Thus in JOHN's method it is $F'(p)\cos\chi$ and $F'(p)\sin\chi$ which are naturally harmonic, whereas in that demonstrated above, $\{F'(p)-\mu p\}\cos\chi$ and $\{F'(p)-\mu p\}\sin\chi$ are forced to be conjugate harmonics. JOHN's method continues in noting that $x_i, i = 1,2$ may be decomposed quite generally in the form

$$x_1 = \phi,_1 + \psi,_2 \quad , \quad x_2 = \phi,_2 - \psi,_1 \qquad 3.5.7$$

where ϕ and ψ are scalar potentials. The solution of 3.5.5 may then be reduced to the solution of two Poisson equations, namely

$$\nabla^2\phi = pA/(A^2+B^2)^{\frac{1}{2}}$$

and 3.5.8

$$\nabla^2\psi = pB/(A^2+B^2)^{\frac{1}{2}} \ .$$

These are then solved using complex variable theory.

It should be noted that JOHN's method for radially symmetric problems produces the same solution as in Section 2.6, and as above, as would be expected. The function p is again constant but that is not apparent from JOHN's formulation, the method of Section 3.2 having the advantage here. However, for Class I materials, JOHN's method is more general in that it makes no a'priori assumptions of the solution. On the other hand, the method of Section 3.2 has the potential of dealing with a larger class of materials in solving problems within the discipline of plane strain finite deformation elasticity, albeit in a cumbersome fashion.

CHAPTER 4 A COMPLEX VARIABLE FORMULATION - GENERAL SOLUTION

SECTION 4.1 INTRODUCTION

In this chapter a complex variable formulation of the equations of Chapter 2 is introduced and developed. A general solution of the field equations is also generated. The material class considered is the harmonic class of equation 3.4.1. This specialisation to harmonic materials is not entirely necessary, but this class includes the semi-linear material (2.5.19), the simplest notional extension of the classical linear material.

In Section 4.2, the complex variables are introduced and the equations of Chapter 2 are re-formulated in terms of these variables. Section 4.3 contains a further discussion of restrictions to be imposed on the material definition in order that it is physically reasonable. In Section 4.4 a completely general closed form solution is generated for harmonic material subjected to plane strain deformations. The solution is given in terms of two arbitrary functions. In Section 4.5 boundary conditions are discussed, particularly in relation to the determination of the two arbitrary functions. The question as to what restrictions have to be placed on the problem in order that the determination is well posed, is addressed. Finally, a small strain asymptotic analysis is undertaken, resulting in the method of Section 4.4 reducing to that detailed by MUSKHELISHVILI (1963), for small strains.

The fitting of boundary conditions and the solution of problems is left until Chapter 5.

SECTION 4.2 A COMPLEX VARIABLE FORMULATION

The underlying complex variables are introduced here. The position variables in the deformed and undeformed (reference) configurations are

$$x = x_1 + ix_2$$

and 4.2.1

$$\mathcal{Z} = X_1 + iX_2$$

respectively, where x_i and X_i are as defined in Section 1.2.

The deformation may thus be written as

$$x = x(\mathcal{Z},\bar{\mathcal{Z}}), \qquad 4.2.2$$

where $\bar{\mathcal{Z}}$ denotes the complex conjugate of $\mathcal{Z}$. The variables $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ may be treated as though they are independent. Simple complex variable manipulation serves to demonstrate that

$$\frac{\partial x}{\partial \mathcal{Z}} = \tfrac{1}{2}(\alpha_{11} + \alpha_{22} + i(\alpha_{21}-\alpha_{12}))$$

and 4.2.3

$$\frac{\partial x}{\partial \bar{\mathcal{Z}}} = \tfrac{1}{2}(\alpha_{11} - \alpha_{22} + i(\alpha_{12}+\alpha_{21})),$$

where $\underline{\alpha}$ is the deformation gradient, the compatibility equations are assumed to hold. In other words, in deriving 4.2.3, $\underline{\alpha}$ is assumed to be derivable from a deformation field.

Now, referring to equation 2.3.2(a), it may be noted that

$$p = 2/x,_{\mathcal{Z}}/ = 2/\bar{x},_{\bar{\mathcal{Z}}}/ \qquad 4.2.4$$

and

$$q = 2/x,_{\bar{\mathcal{Z}}}/ = 2/\bar{x},_{\mathcal{Z}}/. \qquad 4.2.5$$

The second equality in each of the above two equations may be generated by considering the derivative of $\bar{x}$, or via the relationships

$$\text{(i)} \quad \overline{Y,x} = \bar{Y},_{\bar{x}}$$

4.2.6

$$\text{(ii)} \quad /\bar{Y}/ = /Y/,$$

which are true for any complex function Y of a variable x.

Next, the equilibrium equations of 2.2.1 are considered. These are satisfied identically if the stress potentials h_1 and h_2 are defined such that they satisfy 3.2.14. Then writing

$$h = h_1+ih_2, \qquad 4.2.7$$

h is again a function of $\mathcal{Z}$ and $\bar{\mathcal{Z}}$. Assuming that 3.2.14 holds, simple manipulation results in

$$\frac{\partial h}{\partial z} = \tfrac{1}{2}(S_{11} + S_{22} + i\,(S_{12} - S_{21}))$$

and $\qquad$ 4.2.8

$$\frac{\partial h}{\partial \bar{z}} = \tfrac{1}{2}(S_{22} - S_{11} - i\,(S_{12} + S_{21})) \ .$$

Comparing these with 2.4.2 allows the following to be deduced:

$$p_T = \pm\, 2/h,_{z}/ = \pm 2/\bar{h},_{\bar{z}}/$$

and $\qquad$ 4.2.9

$$q_T = 2/h,_{\bar{z}}/ = 2/\bar{h},_{z}/ \quad ,$$

where p_T and q_T are the similarity invariants of the nominal stress $\underline{S}$. Again, the second equality may be derived from first principles or by noting 4.2.6. The sign of p_T is indeterminate as indicated in 4.2.9(1). This is a consequence of its interpretation as the sum of the principal BIOT stresses. On the other hand p, q and q_T are required to be positive.

Now, consider equation 3.2.15 for A, B, C and D of 3.2.2 in terms of h_1 and h_2. This may be employed to demonstrate that

$$2(A + iB) = (h_{1,1} + h_{2,2} + i\,(h_{1,2} - h_{2,1})) \qquad 4.2.10$$

$$= \frac{\partial \bar{h}}{\partial \bar{z}} \quad , \qquad 4.2.11$$

and that

$$2(C + iD) = -(h_{1,1} - h_{2,2} + i\,(h_{1,2} + h_{2,1}))$$

$$= \frac{\partial h}{\partial \bar{z}} \quad . \qquad 4.2.12$$

Next, take equations 3.2.7, bearing in mind 3.2.9 as defining P_T and Q_T, and multiply equations 3.2.7(3) and (4) by i, and then add to 3.2.7, (1) and (2) in a pair-wise fashion, then this will yield

$$\frac{\partial x}{\partial z} = 2\, P_T \frac{\partial h}{\partial z}$$

and $\qquad$ 4.2.13

$$\frac{\partial x}{\partial \bar{z}} = -2\, Q_T \frac{\partial h}{\partial \bar{z}} \quad .$$

Similarly, 3.3.5, 3.3.6 and 3.3.7 may be employed to obtain the results

$$\frac{\partial h}{\partial z} = 2\, P \frac{\partial x}{\partial z} \qquad 4.2.14$$

and

$$\frac{\partial h}{\partial \bar{z}} = -2\, Q \frac{\partial x}{\partial \bar{z}} \quad . \qquad 4.2.15$$

Taking moduli of these last two pairs of equations, noting 4.2.4, 4.2.5, 4.2.9 and 4.2.10 and invoking the definition of the pairs P_T, Q_T (3.2.9) and P, Q (3.3.6), the following relationships may be recovered.

$$Wc_{p_T} = \tfrac{1}{2}p \quad , \quad W_p = \tfrac{1}{2}p_T$$
$$Wc_{q_T} = \tfrac{1}{2}q \quad , \quad W_q = \tfrac{1}{2}q_T \quad . \qquad 4.2.16$$

From the above, equations 4.2.13, 4.2.14 and 4.2.15, the relationships

$$PP_T = \tfrac{1}{4} \quad , \quad QQ_T = \tfrac{1}{4} \qquad 4.2.17$$

are self evident.

The set of equations given above are valid if the equilibrium equations and compatibility conditions are satisfied since they are properties of the associated solution fields. That is, they apply to variables which correspond to an admissible solution of a plane strain, finite deformation, elastostatic problem.

The conditions which must be satisfied in order for the equations to be true are investigated at this stage. Consider the compatibility equations as given in 3.2.12. Employing the identities

$$(\quad)_{,1} \equiv (\quad)_{,z} + (\quad)_{,\bar{z}}$$

and

$$(\quad)_{,2} = i(\quad)_{,z} - i(\quad)_{,\bar{z}}$$

on 3.2.12(i) and (ii) in turn and re-arranging, 4.2.11 and 4.2.12 may be employed in order to obtain the results

$$(P_T\bar{h}_{,\bar{z}})_{,\bar{z}} + (Q_T\bar{h}_{,z})_{,\bar{z}} = 0$$

and 4.2.18

$$(P_T h_{,z})_{,\bar{z}} + (Q_T h_{,\bar{z}})_{,z} = 0 \quad .$$

Noting the identity 4.3.6, these equations may be seen to be mutually conjugate and hence either may be adopted as the compatibility equation. An alternative interpretation is that either of these equations may be regarded as being the necessary condition required to ensure the integrability of 4.2.13. The corresponding condition to ensure integrability of equations 4.2.14 and 4.2.15 is similarly determined, and is

$$(Px_{,z})_{,\bar{z}} + (Qx_{,\bar{z}})_{,z} = 0 \quad , \qquad 4.2.19$$

the equilibrium equation.

The similar functional form of equations 4.2.18(2) and 4.2.19 is yet a further example of the duality pointed out in Sections 2.4 and 3.3. This duality is independent of any assumptions regarding the material definition. The solution of a problem may thus be obtained in two alternative ways:-

(i) A stress potential field h $(z,\bar{z})$ may be presumed to exist. Equations 4.2.18 may then be employed to suitably restrict this function such that 4.2.13 may be integrated to obtain x $(z,\bar{z})$. Now, 4.2.18 involves both P_T and Q_T which, as defined, are functions of p_T and q_T and which are indeed related to h $(z,\bar{z})$. Hence, strictly speaking, an h satisfying 4.2.18 can be found.

(ii) A deformation field x $(z,\bar{z})$ may be presumed to exist. Equation 4.2.19 may be employed to suitably restrict this function such that 4.2.11 and 4.2.12 may be integrated to obtain h $(z,\bar{z})$. This time the differential equation involves P and Q which via p and q are functions of x $(z,\bar{z})$, or at least its derivatives.

In both of the above methods, boundary equations must be imposed as and when practical.

Practically speaking, methods (i) and (ii) are not identical, for the imposition and character of the boundary data may bias the selection of either. Also, it is conventional to specify a material by its strain energy density function W, rather than Wc, the complementary energy density. As such, method (i) may be awkward to apply, in that $P_T(=Wc_{,p_T}/p_T)$ and $Q_T(=Wc_{,q_T}/q_T)$ may not be obtainable as explicit functions of p_T and q_T (Wc is defined via the symbolic Legendre transform 2.4.14). Additionally, any mode of solution involving p_T has the unresolved sign of equation 4.2.9 to contend with. This unresolved sign is not a problem of any great significance, as the indeterminacy is resolved by the boundary data but it is another factor to be borne in mind. In all then, method (ii) would appear a more attractive proposition than (i) for practical purposes.

Rather than discuss boundary conditions at this point, it will be deferred to Section 4.4 as after a general solution has been found such a discussion will be more meaningful.

SECTION 4.3 FURTHER DISCUSSION OF INEQUALITIES: RESTRICTED TO HARMONIC MATERIALS

In order to obtain an analytic solution to the equations of Section 4.2 it is essential that equations 4.2.13 (4.2.19), 4.2.15 and 4.2.16 (4.2.18) are in such a form that they may be integrated analytically to obtain $x\,(z,\bar{z})$ and $h\,(z,\bar{z})$. Alternatively, assumptions as to the form of h or x could suffice, as in Chapter 3, h was assumed to be analytic as a function of z only. In that chapter the technique was shown to be of restricted applicability.

In Chapter 3, two classes of material are discussed, for which the integration to find x and h can be done. The analysis of pertinent inequalities is similar for both classes and is continued with the potentially most useful class, that of the harmonic materials

$$W = f(p) + \tfrac{1}{2}\mu q^2 \qquad 4.3.1$$

As discussed in Chapters 2 and 3, this material was first introduced in F. JOHN (1960), although different variables were used. For ease of reference here the following will be included; some inequalities (and equalities) determined in 2.5, and restricted, where appropriate, to the form 4.3.1:

(i) $f''(2) = \lambda+\mu$, 4.3.2

where λ and μ are the classical Lamé moduli.

(ii) $f(2) = 0,\ f'(2) = 0$ 4.3.3

(iii) $\mu > 0,\quad f''(p) > 0$ 4.3.4

(iv) $f'(p) \gtreqless 0$ as $p \gtreqless 2$. 4.3.5

The harmonic material in the form in 4.3.1, has been little used generally, except in recent papers by KNOWLES & STERNBERG (1975) and papers based on the subject matter of this thesis. JOHN (1960) and (1966) appear to be the only explicit references. The inequalities and limiting properties spelt out in the cited papers will be discussed below since other authors in using the harmonic material, have specified it differently.

Restricting the discussion to materials as defined by 4.3.1 it may then be noted that

$$P = \frac{W_p}{p} = \frac{f'(p)}{p} \qquad 4.3.6$$

and

$$Q = \frac{W_q}{q} = \mu \text{ only.} \qquad 4.3.7$$

Thus it can be concluded from 2.4.14 that

$$\tfrac{1}{2}p_T = f'(p)$$

and $\qquad$ 4.3.8

$$\tfrac{1}{2}q_T = \mu q.$$

Now, given a pure dilation ($\lambda_1 = \lambda_2$) it is reasonable to expect $W \to \infty$ as $J = \lambda_1\lambda_2 \to 0$, where J is, as defined in 1.2.2, the ratio of final to initial volume (surface area in the case of plane strain). It follows that in this case $f(p) \to \infty$ as $p \to 0$, or equivalently

$$f(p) \sim m\ p^{-n}$$

as $p \to 0$ with m and n positive constants. From 4.3.8 it follows that $p_T \to -\infty$ as $p \to 0$ and, moreover, that $f'(p)/p \to -\infty$. This condition was imposed by JOHN (1960) and it implies that the (plane) hydrostatic Cauchy pressure tends to infinity as the volume reduces to zero. On the other hand, KNOWLES & STERNBERG (1975) required that

$$f'(p)/\mu p \to 1$$

as $p \to \infty$. In view of 4.3.8(i), $p_T \to \infty$ as $p \to \infty$. Therefore, invoking 4.3.4, it may be concluded that the first of 4.3.8 is uniquely invertible.

Consideration of a hydrostatic stress, as was done in Section 2.5 results in equation 2.5.11 which may be rewritten as follows

$$pf'(p) - f(p) > 0 \quad , \ p > 0 . \qquad 4.3.9$$

This was required by both JOHN (1960) and KNOWLES & STERNBERG (1975b) on identical grounds. It is to be noted that it is automatically satisfied for $p \leq 2$ by 4.3.4 and 4.3.5.

Then coupling the requirement $f'(p)/\mu p \to 1$ as $p \to \infty$ with 4.3.9.

$$f'(p)/\mu p < 1 \quad \forall\, p > 0 \qquad 4.3.10$$

is obtained. This inequality will be considered further when solving a particular problem as it will be found to be insufficiently strong.

Now, from the fact that $f'(p)$ is a strictly monotonic increasing function of p and that $f'(p) \to -\infty$ as $p \to 0$, it is clear that the equation

$$f'(p) + \mu p = 0 \qquad 4.3.11$$

has a unique solution p_o say, this is because $-\mu p$ is monotonically decreasing and is greater than $f'(p)$ at $p = 0$. In addition, as $f'(2) = 0$, it follows that

$$0<p_o<2 \ . \qquad 4.3.12$$

This condition has been strengthened by KNOWLES & STERNBERG (1975b) to

$$1<p_o<2 \ ,$$

the argument being on physical grounds.

The root p_o of 4.3.11 is of great importance in JOHN's analysis, and both JOHN and KNOWLES & STERNBERG restrict their analysis to deformations such that $p>p_o$ everywhere. For $p<p_o$ the direction of maximal strain is not coaxial with the direction of maximal Cauchy stress. The authors deemed this to be inadmissible and hence imposed the restriction.

The above discussion summarises the restrictions placed on $f(p)$ by the various authors. The various conclusions and results will be further discussed as they arise in the current text. For the immediate purpose the only requirement is that the results of Section 2.5 hold, and that the limit conditions

$$f'(p) \to -\infty \text{ as } p \to 0$$

and $\qquad 4.3.13$

$$f'(p) \to \infty \text{ as } p \to \infty \ ,$$

are true. These are really just statements regarding the nominal hydrostatic pressure at extreme deformations.

SECTION 4.4 A GENERAL SOLUTION FOR A CLASS OF MATERIALS

In this section the complex field equations developed in Section 4.2, for the material class explicitly selected in Section 4.3, are solved.

Substituting 4.3.6 and 4.3.7 into 4.2.14 and 4.2.15 respectively, the following are obtained

$$\frac{\partial h}{\partial \bar{z}} = 2\,P\,\frac{\partial x}{\partial \bar{z}}$$

and 4.4.1

$$\frac{\partial h}{\partial z} = -2\mu\,\frac{\partial x}{\partial z}\ ,$$

where P is a function of $p = 2/\partial x/\partial \bar{z}/$. Correspondingly, 4.2.13 becomes

$$\frac{\partial x}{\partial \bar{z}} = 2\,P_T\,\frac{\partial h}{\partial \bar{z}}$$

and 4.4.2

$$\frac{\partial x}{\partial z} = -\frac{1}{2\mu}\,\frac{\partial h}{\partial z}$$

where $P_T = p/4f'(p)$.

Note that $\frac{1}{2}p_T = W_p = f'(p)$ for this material and $f'(p)$ is strictly monotonic and hence simple valued; consequently $p = (f')^{-1}(p_T/2)$ is well defined.

The second equations in each of 4.4.1 and 4.4.2 are equivalent. Each may be integrated directly to yield

$$h = -2\mu x + 2\mu g(\bar{z}), \qquad 4.4.3$$

where $g(\bar{z})$ is an arbitrary function. Using this to eliminate h from 4.4.1(1) results in

$$2\mu g'(\bar{z}) = 2(P+\mu)\,\frac{\partial x}{\partial \bar{z}}\ , \qquad 4.4.4$$

while eliminating x between 4.4.3 and 4.4.2(1) results in

$$g'(\bar{z}) = 2\,(P_T + \tfrac{1}{4}\mu)\,\frac{\partial h}{\partial \bar{z}}\ . \qquad 4.4.5$$

Now, taking the modulus of 4.4.4 and using 4.2.4 to eliminate $/\partial x/\partial \bar{z}/$ and remembering that $Pp = W_p$ (3.3.6), the following is obtained;

$$\pm\ 2\mu/g'(\bar{z})/ = f'(p) + \mu p. \qquad 4.4.6$$

It is convenient that $\bar{\Phi}(p)$ is introduced such that

$$2\mu\bar{\Phi}(p) = f'(p) + \mu p \equiv 2\mu F'(p) \qquad 4.4.7$$

where F(p) is the function of p as used by JOHN (1960) (see 2.3.18). Then 4.4.6 becomes

$$\bar{\Phi}(p) = \pm\, |g'(z)| \ . \qquad 4.4.8$$

In Section 4.3 the equation $\bar{\Phi}(p) = 0$ was considered and it was concluded that it has a root $p_o (1 < p_o < 2)$. The importance other authors have placed on p_o was also pointed out. Now, 4.4.7 indicates that $\bar{\Phi}(p) = 0$ when $|g'(z)|$ is zero but this can only occur at isolated points, provided $g'(z) \neq 0$ over its domain of analyticity. Indeed, the Maximum Modulum theorem indicates that $\bar{\Phi}(p) = 0$ can only occur at the boundary of the domain. The contention is, and this will be borne out in the next chapter, that the sign employed in 4.4.6 is a function only of the boundary data. The sign is constant throughout a body for a given set of boundary conditions. This merely confirms the remarks of JOHN (1960) to the effect that F'(p) has a constant sign independent of z.

As in deriving 4.4.6 the modulus of 4.4.5 may be taken and employed to obtain

$$\pm\, |g'(z)| = (f')^{-1}(p_T/2) + p_T/4\mu \ . \qquad 4.4.9$$

In writing

$$\Psi(p_T) = (f')^{-1}(p_T/2) + p_T/4\mu \qquad 4.4.10$$

it can be seen, as would be expected, that 4.4.6 and 4.4.9 are equivalent. This occurs since

$$\begin{aligned}\Psi(p_T) &= \Psi\{2f'(p)\} = (f')^{-1}(f'(p)) + 2f'(p)/4\mu \\ &= p + \frac{f'(p)}{2\mu} = \bar{\Phi}(p) \ ,\end{aligned}$$

where f'(p) has been relied upon to be strictly monotonic in order to guarantee that $(f')^{-1}$ is well defined.

Now, from 4.4.8 it is inferred that p may be determined as a function of z and $\bar{z}$ once $g'(z)$ has been determined. Additionally, given f'(p) is monotonic, $\bar{\Phi}$ is also monotonic and hence

$$p = \bar{\Phi}^{-1}\,(\pm |g'(z)|) \ , \qquad 4.4.11$$

formally. The inverse $(f')^{-1}(p_T/2)$ must also be strictly monotonic and hence so must $\Psi(p_T)$. Consequently,

$$p_T = \gamma^{-1}(\pm/g'(\bar{z})/)$$

may be written as the well-defined inverse to 4.4.9. The critical value p_{To} at which 4.4.10 changes sign is given by $p_{To} = 2f'(p_o)$. It is negative and it corresponds to a net pressure.

Consider 4.4.4, where

$$\frac{\partial x}{\partial \bar{z}} = \frac{\mu g'(\bar{z})}{(P+\mu)} = \frac{\mu g'(\bar{z})p}{(f'(p)+\mu p)} \quad .$$

Then, on employing 3.3.6(1), it in turn becomes

$$\frac{\partial x}{\partial \bar{z}} = \pm \bar{\Phi}^{-1} \frac{(\pm/g'(\bar{z})/)g'(\bar{z})}{2/g'(\bar{z})/} \quad , \qquad 4.4.12$$

where 4.4.6 and 4.4.11 have been used to eliminate the terms involving p.

Similarly, 3.4.14, 4.4.10 and 4.4.12 may be used to eliminate P_T from 4.4.5 to obtain

$$\frac{\partial h}{\partial \bar{z}} = \pm \gamma^{-1} \frac{(\pm/g'(\bar{z})/)g'(\bar{z})}{2/g'(\bar{z})/} \quad . \qquad 4.4.13$$

The right-hand sides of 4.4.12 and 4.4.13 are known functions of the arbitrary $g'(\bar{z})$ and its conjugate, thus these equations may be integrated to yield

$$x = \pm\tfrac{1}{2} \int^{\bar{z}} \bar{\Phi}^{-1} \frac{(\pm/g'(\bar{z})/)g'(\bar{z})d\bar{z}}{/g'(\bar{z})/} + K(\bar{\bar{z}}) \qquad 4.4.14$$

and

$$h = \pm \int^{\bar{z}} \gamma^{-1} \frac{(\pm/g'(\bar{z})/)g'(\bar{z})d\bar{z}}{/g'(\bar{z})/} + K^*(\bar{\bar{z}}) \quad , \qquad 4.4.15$$

where $K(\bar{\bar{z}})$ and $K^*(\bar{\bar{z}})$ are arbitrary. The integrations are with respect to $\bar{z}$, with $\bar{\bar{z}}$ fixed.

Thus far there has been some redundancy in the development of the theory in this section, since the arguments have been duplicated at all points in order to highlight, and is facilitated by the elegant and robust duality of the formalism.

The equations 4.2.18 and 4.2.19 have been solved in a parallel fashion. The solution of either of these equations would suffice, yielding 4.4.14 or 4.4.15 respectively, since 4.4.3 enables an expression for h to be obtained from 4.4.14, and for x to be obtained from 4.4.15. In going from 4.4.14 to an expression for h, using 4.4.3 allows

$$\frac{h}{2\mu} = \mp \int^{\bar{z}} \frac{\bar{\Phi}^{-1}(\pm/g'(\bar{z})/)g'(z)d\bar{z}}{/g'(z)/} + g(z) - K(\bar{z}), \qquad 4.4.16$$

to be obtained.

Consistency of 4.4.16 and 4.4.15 is ensured provided

$$K^*(\bar{z}) = -2\mu K(\bar{z})$$

and 4.4.17

$$\Psi^{-1}(\pm/g'(z)/) = \pm 4\mu/g'(z)/ - 2\mu\, \bar{\Phi}^{-1}(\pm/g'(z)/).$$

The complete closed form solution is thus obtained.

Now, 4.4.17 may be rewritten using 4.4.11 and 4.4.12 as

$$p_T = \pm 4\mu/g'(z)/ - 2\mu p \;.$$

This result is known to be true from an argument based on other results. Using 4.4.6 this equation may be reduced to

$$p_T = 2f'(p) + 2\mu p - 2\mu p,$$

which is seen to be an identity when compared with 2.4.14. Reversibility of the above steps is sufficient to ensure that the solutions 4.4.14 and 4.4.15 are consistent - with the proviso that 4.4.17 is true. In subsequent work presented in this thesis 4.4.14 and 4.4.16 are adopted to represent the general solution for the plane strain finite deformation elastostatic problem for harmonic materials. It is to be noted that either the upper or lower sign must be taken throughout the adopted solution pair, which in turn depends on the boundary conditions. The determination of the complete solution depends on using these boundary conditions to determine the as yet arbitrary $g'(z)$ and $K(\bar{z})$.

At this point the determination of $g'(z)$ and $K(\bar{z})$ from the boundary data will be discussed. Firstly, one simple smooth boundary which is specified as

$$z = z_1(\bar{z}) \quad , \qquad 4.4.18$$

will be considered. Secondly a pair of such boundaries will be considered. An example of a boundary of this type is $z = A^2/\bar{z}$ which defines a circle of radius A. This form will be used extensively in the next chapter when specific solutions are considered, conformal transformations could be employed to map a more general boundary on to a circle. For reasons that will become apparent, the notation

$$I[\lambda,\mu] = \int_{\mu}^{\lambda} \bar{\Phi}^{-1}\left(\pm[g'(\xi)]^{\frac{1}{2}}[\bar{g}'(\bar{z})]^{\frac{1}{2}}\right)[g'(\xi)]^{\frac{1}{2}}\,d\xi, \qquad 4.4.19$$

is introduced. If the integral is indefinite the representation

$$I[\lambda]$$

will be employed.

Consider firstly the boundary condition of place imposed along the simple curve 4.4.18, represented by

$$\hat{x}_1 = x_1(z), \qquad 4.4.20$$

where $x_1(z)$ is the prescribed deformation and $\hat{x}_1$ is the restriction of x to the contour 4.4.18. For simple smooth contours this specification is quite general. Consider a parametrized curve defined by

$$z = f(\alpha) \qquad \alpha \in [\alpha_B, \alpha_T] \quad ,$$

where α is real and neither α_B nor α_T is constrained to be finite. The simplicity of the curve ensures that

$$f(\alpha_1) \neq f(\alpha_2) \quad \forall \alpha_1, \alpha_2 \in [\alpha_B, \alpha_T) \;, \; \alpha_1 \neq \alpha_2 \;.$$

Consequently

$$\alpha = f^{-1}(z) \qquad\qquad (*)$$

is well defined, as is the function

$$\alpha = \bar{f}^{-1}(\bar{z}).$$

These may be employed to eliminate α and to obtain

$$f^{-1}(z) = \bar{f}^{-1}(\bar{z}),$$

hence

$$z = f[\bar{f}^{-1}(\bar{z})].$$

Additionally, assume that the boundary condition is specified as

$$x_1{}^*(\alpha) \qquad \alpha \in [\alpha_B, \alpha_T)$$

then (*) may be employed to eliminate α and obtain

$$x_1(z) = x_1{}^*(f^{-1}(z)).$$

This serves to define $x_1(z)$. Suitably equating this to the restriction of the general x to the contour $\hat{x}_1$, the representation 4.4.20 is obtained.

Returning to the consideration of the boundary condition of place, equation 4.4.14 may be written as

$$x = \pm\tfrac{1}{2}[g'(z)]^{-\frac{1}{2}}\int^{z} \Phi^{-1}(\pm[g'(\zeta)]^{\frac{1}{2}}[g'(\bar{z})]^{\frac{1}{2}})[g'(\zeta)]^{\frac{1}{2}}\, d\zeta + K(\bar{z}) \qquad 4.4.21$$

which, taking into account the definition 4.4.19, becomes

$$x = \pm\tfrac{1}{2}[g'(\bar{z})]^{\frac{1}{2}}\ I[z] + K(\bar{z}). \qquad 4.4.22$$

Combining this with 4.2.20 yields

$$x_1(z) = \pm\tfrac{1}{2}[g'(\bar{z})]^{-\frac{1}{2}}\ I[z] + K(\bar{z}), \qquad 4.4.23$$

which is evaluated for all z such that 4.4.18 holds. In other words

$$x_1(z_1(\bar{z})) = \pm[g'(\bar{z})]^{-\frac{1}{2}}\ I[z_1(\bar{z})] + K(\bar{z}) \qquad 4.4.24$$

which may be re-arranged to obtain $K(\bar{z})$ as follows:-

$$K(\bar{z}) = x_1(z_1(\bar{z})) \mp \tfrac{1}{2}[g'(\bar{z})]^{-\frac{1}{2}}\ I[\bar{z}_1(\bar{z})]. \qquad 4.4.25$$

As written in 4.4.25 the function $K(\bar{z})$ is defined only for all $\bar{z}$ such that the inverse of 4.4.18 holds. However, 4.4.25 is an identity in $\bar{z}$ and as such $K(\bar{z})$ may be extended to all points $\bar{z}$ in the body. The elimination of $K(\bar{z})$ between 4.4.25 and 4.4.22 yields,

$$x = x_1(z_1(\bar{z})) \pm [g'(\bar{z})]^{-\frac{1}{2}}\ I[\bar{z}, z_1(\bar{z})]. \qquad 4.4.26$$

It follows that x can be seen to depend on $g'(\bar{z})$ only. The corresponding form for h may be determined from 4.4.26 and 4.4.3 to yield

$$h/_{2\mu} = g(z) - x_1\{z_1(\bar{z})\} \mp \tfrac{1}{2}[g'(\bar{z})]^{-\frac{1}{2}}\ I[\bar{z}, z_1(\bar{z})]. \qquad 4.4.27$$

Consequently it may be seen that $K(\bar{z})$ has been eliminated from 4.4.14 and 4.4.16. in favour of a boundary condition of place on a simple smooth boundary as specified in 4.4.18.

Consider now a traction boundary condition, again applied on a curve as in 4.4.18. By denoting the normal to any curve of the form 4.4.18 by N_1 and N_2 and the nominal traction on this curve by t_1 and t_2, and by invoking 1.3.3 this produces the result,

$$t_1 = S_{11}N_1 + S_{21}N_2\ , \quad t_2 = S_{12}N_1 + S_{22}N_2, \qquad \mathbf{4.4.28}$$

where $\underline{S}$ is the nominal stress. Then, writing

$$t = t_1 + it_2\ , \qquad N = N_1 + iN_2$$

the following is obtained using 4.2.8

$$t = N\frac{\partial h}{\partial z} - \bar{N}\frac{\partial h}{\partial \bar{z}}. \qquad 4.4.29$$

Adopting the form of curve given in 4.4.18 and writing it in the form

$$\mathcal{F}(z,\bar{z}) \equiv z - z_1(\bar{z}) = 0$$

it is possible to use the result that the tangent to the curve is given by

$$\frac{i\mathcal{F}_{,\bar{z}}}{|\mathcal{F}_{,\bar{z}}|}. \qquad 4.4.30$$

By employing this, an expression for T, the complex tangent, in terms of z, as

$$T = \left(\frac{\partial z_1}{\partial \bar{z}}\right)^{\frac{1}{2}} \left\{ = \left(\frac{d\bar{z}_1}{\partial Z}\right)^{-\frac{1}{2}} \right\} \qquad 4.4.31$$

may be obtained. Consequently, an expression for the complex normal N results as follows:

$$N = -iT = -i\left(\frac{d\bar{z}_1}{dz}\right)^{-\frac{1}{2}} \qquad 4.4.32$$

Hence, using 4.4.32 and 4.4.29 with the second equality in 4.4.31

$$t = -i\left(\frac{d\bar{z}_1}{dz}\right)^{-\frac{1}{2}} \frac{d\hat{h}_1}{dz} \qquad 4.4.33$$

is obtained, where $d\hat{h}_1/dZ$ is the directional (path) derivative of

$$\hat{h}_1(z) = h(z,\bar{z}_1(z)).$$

The boundary condition of traction can be written

$$t = \hat{t}_1(z) \qquad 4.4.34$$

on the boundary $z = z_1(\bar{z})$, where $\hat{t}$ is specified. This result may be inserted into 4.4.33, which upon integration yields

$$\hat{h}_1 = h_1(z), \qquad 4.4.35$$

where $h_1(z)$ is known apart from an additive constant.

Thus it can be concluded that, apart from a constant, the specification of the nominal traction on a smooth simple contour is equivalent to specifying the stress potential h on that contour. For the remainder of this thesis the form 4.4.35 is adopted at the traction boundary condition. The adoption of this result has the pleasing consequence of continuing the duality in that the forms for x and h are similar in structure.

Proceeding, then, as in deriving 4.4.26 from 4.4.14, again, the arbitrary function $K(\bar{z})$ is eliminated from 4.4.16, given 4.4.35 as specified on some contour as 4.4.18, to obtain

$$h = h_1\left(z_1(\bar{z})\right) + 2\mu[g(z)-g(z_1(\bar{z}))] \mp \mu\, [g'(\bar{z})]^{-\frac{1}{2}}\, I\, [z, z_1(\bar{z})], \qquad 4.4.36$$

where $I\,[\ ,\ ]$ is as defined in 4.4.19. The corresponding form for x may be recovered from 4.4.3 as

$$2\mu x = 2\mu g\left(z_1(\bar{z})\right) - h_1\left(z_1(\bar{z})\right) \pm \mu\, [\bar{g}'(\bar{z})]^{-\frac{1}{2}}\, I\, [\bar{z}, z_1(\bar{z})]. \qquad 4.4.37$$

Thus, both the traction and deformation boundary conditions as applied on a simple contour have been tackled. It is to be noted that in each case the arbitrary $K(\bar{z})$ was eliminated. The boundary conditions specified on a second contour are now considered. The aim being to investigate how these may be utilized to determine $g'(z)$.

The contour is taken to be specified in the same form as 4.4.18, specifically

$$z = z_2(\bar{z}) \text{ or } \bar{z} = \bar{z}_2(z). \qquad 4.4.38$$

A boundary condition of place is specified as

$$\hat{x}_2 = x_2(z), \qquad 4.4.39$$

where $x_2(z)$ is prescribed, and $\hat{x}_2(z) = x(z, \bar{z}_2(z))$ as in 4.4.20. A traction boundary condition is taken in the form

$$\hat{h}_2 = h_2(z), \qquad 4.4.40$$

where h_2 is prescribed and $\hat{h}_2(z) = h(z, \bar{z}_2(z))$. The following notation is introduced

$$J\,[\lambda,\mu] = \int_{\mu}^{\lambda} \bar{\Phi}^{-1} \left(\pm [g'(\zeta)]^{\frac{1}{2}} [\bar{g}'\left(\bar{z}_2(z)\right)]^{\frac{1}{2}}\right) [g'(\zeta)]^{\frac{1}{2}}\, d\zeta. \qquad 4.4.41$$

It can easily be seen to be the integral $I[\lambda, \mu]$, when $\bar{z}$ is replaced by $\bar{z}_2(z)$.

Firstly, consider the solution 4.4.26 which is such that a boundary condition of place has been satisfied on one contour. This result is employed to determine $g(z)$ when a second condition of place on a second contour is specified. Now, 4.4.26 is valid for all z contained in the body, in particular on the second contour 4.4.38 where 4.4.39 is specified, in which case 4.4.39 may be employed and $\bar{z}$ constrained as $\bar{z} = \bar{z}_2(z)$ to give

$$x_2(\mathfrak{z}) = x_1\left\{\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right]\right\} \pm \tfrac{1}{2}\left[\bar{g}'\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right]^{-\frac{1}{2}} \Upsilon\left[\bar{\mathfrak{z}}, \mathfrak{z}_1\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right] \qquad 4.4.42$$

This theoretically allows $g'(\mathfrak{z})$ to be determined but it must not be overlooked that $\Upsilon[\ ,\ ]$ involves $g'(\mathfrak{z})$. It may be argued that 4.4.42 allows $g'(\mathfrak{z})$ to be determined on $\mathfrak{z} = \mathfrak{z}_2(\bar{\mathfrak{z}})$ only, but as it is an identity, any solution may be extended over the interior of the body (between the boundaries).

Secondly, and in a similar fashion, 4.4.37 may be employed with the condition of place 4.4.39 to determine $g(\mathfrak{z})$ as the solution of the following

$$2\mu x_2(\mathfrak{z}) = 2\mu g\left\{\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right]\right\} - h_1\left\{\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right]\right\} \pm$$
$$\pm \mu\left[\bar{g}'\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right]^{-\frac{1}{2}} \Upsilon\left[\bar{\mathfrak{z}}, \mathfrak{z}_1\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right] \qquad 4.4.43$$

Thirdly, using 4.4.27 and 4.4.40 the expression

$$\frac{h_2(\mathfrak{z})}{2\mu} = g(\mathfrak{z}) - x_1(\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right]) \mp$$
$$\mp \tfrac{1}{2}\left[\bar{g}'\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right]^{-\frac{1}{2}} \Upsilon\left[\bar{\mathfrak{z}}, \mathfrak{z}_1\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right] \qquad 4.4.44$$

is obtained, from which $g(\mathfrak{z})$ may be determined. This case is that where a traction boundary condition is specified on the second contour and one of place on the first.

Finally, 4.4.36 and 4.4.40 permit the generation of an expression from which $g(\mathfrak{z})$ may be determined, where two traction boundary conditions are specified. That expression is

$$h_2(\mathfrak{z}) = h_1\left\{\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right]\right\} + 2\mu\left[g(\mathfrak{z}) - g(\mathfrak{z}_1\left[\bar{\mathfrak{z}}_2(\mathfrak{z})\right])\right] \mp$$
$$\mp \mu\left[\bar{g}'\left\{\bar{\mathfrak{z}}(\mathfrak{z})\right\}\right]^{-\frac{1}{2}} \Upsilon\left[\bar{\mathfrak{z}}, \mathfrak{z}_1\left\{\bar{\mathfrak{z}}_2(\mathfrak{z})\right\}\right] . \qquad 4.4.45$$

Each of the above four equations may be used to determine $g'(\mathfrak{z})$. Thus a solution may be determined.

Prior to discussing the uniqueness and existence of the above solutions, the extent to which the equations simplify for the semi-linear material is investigated

$$W = \frac{\lambda+\mu}{2}(p-2)^2 + \frac{\mu}{2} q^2 \qquad 4.4.46$$

is first introduced as 2.5.19, but is repeated here for ease of reference.

Firstly, note that $\bar{\Phi}(p)$ of 4.4.7 is zero for

$$p = p_o = 2(\lambda+\mu)/(\lambda+2\mu) \quad . \tag{4.4.47}$$

Where p_o, being the critical value deciding which sign is to be taken throughout this chapter, is the important quantity discussed after equation 4.4.8. If the boundary conditions are such that $p > p_o$, the top sign must be used, otherwise the lower must be taken.

From 4.4.7 and 4.4.11 and assuming 4.4.46 as a material definition, it follows that

$$p = \bar{\Phi}^{-1}(\pm/g'(z)/) = \frac{2(\lambda+\mu)}{\lambda+2\mu} \pm \frac{2\mu/g'(z)/}{\lambda+2\mu} . \tag{4.4.48}$$

Equation 4.4.14 then becomes

$$x = \pm \frac{1}{2}\int \frac{2(\lambda+\mu) \pm 2\mu(g'(\zeta))^{\frac{1}{2}}(\bar{g}'(\bar{z}))^{\frac{1}{2}}}{(\lambda+\mu)\ (g'(\zeta))^{\frac{1}{2}}(\bar{g}'(\bar{z}))^{\frac{1}{2}}}\ g'(\zeta)d\zeta + K(\bar{z})$$

which simplifies to

$$x = \frac{\mu}{(\lambda+2\mu)}\ g(z) \pm \frac{\lambda+\mu}{(\lambda+2\mu)(\bar{g}'(\bar{z}))^{\frac{1}{2}}} \int^{z} g'(\zeta)^{\frac{1}{2}}d\zeta + K(\bar{z}) . \tag{4.4.49}$$

Employing 4.4.3 the corresponding stress potential h, may be determined as

$$\frac{h}{2\mu} = \frac{\lambda+\mu}{\lambda+2\mu}\left[g(z) \mp \frac{1}{(g'(z))^{\frac{1}{2}}}\int^{z} g'(\zeta)^{\frac{1}{2}}d\zeta \right] - K(\bar{z}) \quad . \tag{4.4.50}$$

The two equations from which the arbitrary functions $K(\bar{z})$ and $g(z)$ may be determined are quoted here. This is for the case when two boundary conditions of place have been specified. Equation 4.4.25 becomes

$$\begin{aligned} K(\bar{z}) &= x_1(\ z_1(\bar{z})) - \frac{\mu}{\lambda+2\mu}\ g(z,(\bar{z})) \mp \\ &\mp \frac{(\lambda+\mu)}{(\lambda+2\mu)} \int^{z_1(\bar{z})} \frac{g'(\zeta)^{\frac{1}{2}}}{(\bar{g}'(\bar{z}))^{\frac{1}{2}}}\ d\zeta \quad , \end{aligned} \tag{4.4.51}$$

and 4.4.42 becomes

$$\begin{aligned} x_2(z) &= x_1\left(z_1\left[\bar{z}_2(z)\right]\right) + \\ &+ \frac{\mu}{\lambda+2\mu}\left(g(z) - g(z_1(\bar{z}_2(z)))\right) \pm \\ &\pm \int_{z_1(\bar{z}_2(z))}^{z} \frac{(\lambda+\mu)}{(\lambda+2\mu)} \frac{g'(\zeta)^{\frac{1}{2}}}{\bar{g}'(\bar{z}(z))^{\frac{1}{2}}}\ d\zeta . \end{aligned} \tag{4.4.52}$$

This particular example has been chosen because physical arguments imply that this situation is the most stable and so the one to which the

The general question of uniqueness is considered now. However, at this moment in time it is to be remarked that the theory is not sufficiently advanced to discuss the general solutions. HILL (1958) and (1968b) has generated sufficient conditions such that the solution of incremental problems be unique. These conditions were discussed in Section 2.5 in the context of a requirement for physically reasonable material behaviour. In terms of the material specified by 4.3.1, the conditions 2.5.6 and 2.5.7 become

$$\mu q \geqslant 0 \quad \text{and} \quad \mu f''(p) > 0 \tag{4.4.53}$$

respectively. These conditions are also sufficient to ensure stability of solution. Thus in general, it can be said that provided 4.4.53 holds, any solution is locally unique. This does not mean that the solution is not independent of the stress/strain history. However, if there exists some neighbourhood $\mathcal{N}$, of p = 2 and q = 0 in invariant space such that 4.4.53 holds, then the result can be interpreted as follows:-

> "If the material configuration was originally in $\mathcal{N}$ and its subsequent strain history has been contained in $\mathcal{N}$ <u>at all points of the material</u>, then the current configuration is unique. As the set $\mathcal{N}$ is open it can also be said that the solution is stable."

Thus the problem that remains is whether any spurious solutions have been introduced by the technique, assuming that the solution has remained within $\mathcal{N}$. Indeed, it is important to consider what, if any, is the redundancy in the pair $(g(\overline{z}),K(\overline{z}))$. Each function $g(\overline{z})$ or $K(\overline{z})$ may contain some redundant information. Any constant term in $g(\overline{z})$ is but a trivial example. The pair may interact such that other additional terms do not affect the overall solution as specified by the stress and deformation fields. From the general form 4.3.1 it is obviously a task of considerable complexity to resolve the question of uniqueness. JOHN (1960) has proved that given 4.4.53, and provided the strain history remains in $\mathcal{N}$, then the solution he generates is unique, provided that the boundary data is that of place only, and provided also that the region of consideration is simply connected.

It is formally straightforward to generalise the technique for determining $g(z)$ and $K(\bar{z})$ employed above, to the case where the material is bounded by a set of piecewise smooth contours. Equations may be written which specify $g(z)$ and $K(\bar{z})$ in this multiply connected region but the manipulation of these equations would prove somewhat troublesome and this extension is omitted here. In discussing the problem of uniqueness for the semi-linear material, however, multiply connected regions are not precluded.

The equations 4.4.49-52 are intimately dependent upon the form

$$\int_{\mu}^{\lambda} g'(\zeta)^{\frac{1}{2}} d\zeta \qquad 4.4.54$$

where λ, μ are various functions of z and $\bar{z}$. The functions $K(\bar{z})$, $x(z,\bar{z})$ and $h(z,\bar{z})$ are well defined if 4.4.54 is path independent. This, itself, depends on the analycity of $g'(\zeta)^{\frac{1}{2}}$, which in turn depends upon the singularities of $g'(\zeta)$. An obvious restriction, it may be thought, is that $g'(\zeta)$ should have no singularities, such that ζ is a point of the material. However, this condition is not sufficient as the region of integration is defined by the union of the following:

$V_o = (z:$ s.t. z is a point of the body$)$

$V_1 = (z = z_2(\bar{z})$: s.t. $\bar{z} \in \bar{V}_o)$

$V_2 = (z = z_1(\bar{z})$: s.t. $\bar{z} \in \bar{V}_o)$

$V_{12} = (z = z_2(\bar{z}_1(z))$: s.t. $z \in V_o)$

$V_{21} = (z = z_1(\bar{z}_2(z))$: s.t. $z \in V_o)$

in other words, $g'(\zeta)$ must be analytic over

$$\zeta \in V > V_o \cup [\bar{V}_1 \cup V_2] \cup [\bar{V}_{12} \cup V_{21}]. \qquad 4.4.55$$

This is a sufficient condition for the integral to be simple valued. In addition provided $g'(\zeta)$ has no zeros for $\zeta \in V_o$, it is also certain that x and h are well defined and finite everywhere.

The fact that $g'(\zeta)^{\frac{1}{2}}$ is by necessity of multiplicity two is not disturbing, as the sign to be used in equations 4.4.49-52 is fixed by the boundary data through 4.4.52 and this will select the correct branch. Unfortunately, the situation is not as simple as the discussion above would imply. In Chapter 6 a solution is continued numerically and the expression 4.5.54 is shown to have two branch points in V, cf 4.5.55. Indeed, for an applied traction greater than a particular value, these branch points correspond to points within the material. The solution fields are fully discussed there.

This section is concluded by repeating that the question of uniqueness, when applied to the technique introduced in this chapter, is as yet unresolved. All indications are that the resolution is not a trivial task but is worthy of note, even for the semi-linear material.

SECTION 4.5 SMALL STRAIN ASYMPTOTICS

In this section the technique developed in Section 4.4, is restricted such that the strains are small. The small strain is assumed at this point, to be characterised by

$$p = (\lambda_1+\lambda_2) = 1 + E, \qquad 4.5.1$$

where /E/ is small.

Consider the equilibrium equations in terms of the Cauchy stress (1.5.4(1)) restricted to plane strain. Employing the argument as used in Section 3.2, η_1 and η_2 are now introduced such that

$$\sigma_{11} = \eta_{2;2}, \quad \sigma_{21} = -\eta_{2;1}, \quad \sigma_{12} = -\eta_{1;2}, \quad \sigma_{22} = \eta_{1;1}, \qquad 4.5.2$$

where ";" has been used to denote differentiation with respect to the co-ordinates in the current configuration. Now $\underline{\sigma}$ is symmetric, hence

$$\eta_{2;1} - \eta_{1;2} = 0 \qquad 4.5.3$$

Consequently the existence of a further function Γ of x_1 and x_2 such that

$$\Gamma_{;2} = \eta_2 \text{ and } \Gamma_{;1} = \eta_1, \qquad 4.5.4$$

may be inferred. Equation 4.5.3 is satisfied automatically. The function Γ is called the AIRY stress function, and is widely used in classical elasticity theory. The major advantage in using Γ is the simple form the equilibrium equations assume once the compatibility equations have been satisfied, which is that of a bi-harmonic equation for Γ. The works of **MUSKHELISHVILI** (1963) are based on the solution of this equation.

Now, consider the relationship 3.2.14, which are in terms of the Langrangian co-ordinates (X_1,X_2). Referring them to the Eulerian pair (x_1,x_2), employing the relationship

$$h_{n,i} = h_{n;j}\ \alpha_{ji}, \qquad 4.5.5$$

and substituting for the nominal stress, via 3.2.4, and employing 1.3.2 and 4.5.2 with 4.5.4, it may be demonstrated that

$$\tfrac{1}{2}h = \frac{\partial\Gamma}{\partial x}.$$

Γ is regarded as a function of x and $\bar{x}$, h is again h_1+ih_2 of 4.2.7. On linearization using the above, 4.2.18 may be shown to be equivalent to the classical

$$\frac{\partial^4 \Gamma}{\partial x^2 \partial \bar{x}^2} = 0 \qquad 4.5.6$$

of the infinitesimal theory. Thus, for small strains, the generated field equations reduce to their linear small strain equivalents.

A more direct comparison with the linear theory is afforded by a linearization of the solution fields presented in Section 4.4. In Section 2.5, a particular member of the class of harmonic materials, the semi-linear material was identified as satisfying all the asymptotic requirements for small strains. This can be rephrased as ; as the strains become small, the material behaviour must approach that exhibited for the semi-linear material. The semi-linear material itself, is the simplest material, such that the small strain analysis is consistent with the linear theory.

In applying the analysis, start from the solution as stated by equation 4.4.14. The form 4.4.50 is inappropriate, insofar as it is not amenable to a small strain analysis. Given

$$W = \frac{\lambda+\mu}{2}(p-2)^2 + \frac{\mu}{2} q^2 \qquad 4.5.7$$

the semi-linear material, then

$$f'(p) = (\lambda+\mu)(p-2). \qquad 4.5.8$$

Substituting this f'(p) into 4.4.4, taking the modulus and employing 4.2.4

$$p = \frac{2(\lambda+\mu) \pm 2\mu/g'(z)/}{(\lambda+2\mu)} \qquad (= \bar{\Phi}^{-1}(/g'(z)/)), \qquad 4.5.9$$

is determined. Inserting this into 4.4.14, the following is obtained;

$$x = \frac{1}{2}\int^{z} \left[\frac{2(\lambda+\mu) + 2\mu/g'(z)/}{(\lambda+2\mu)} \right] \frac{g'(z)}{/g'(z)/} dz + K(\bar{z}) \qquad 4.5.10$$

where the top (+ve) sign has been taken throughout. The reason for taking this sign is that small strains are to be considered consequently it is ensured that $p>p_0>0$.

In contrast to 4.5.1 a small strain specification of

$$g'(z) = 1 + \eta(z) \qquad 4.5.11$$

is adopted, where $|\eta(z)|$ is uniformly small over the body. The asymptotic analysis consists of a linearization with respect to $\eta(z)$ as a small quantity. Given $g'(z)$ as above, yields

$$|g'(z)| = 1 + \eta(z) + \overline{\eta}(z)$$

and 4.5.12

$$|g'(z)|^{-1} = 1 - \eta(z) - \overline{\eta}(z)$$

to first order in $|\eta(z)|$. Assuming the terms of second order in $|\eta(z)|$ are zero, and inserting 4.5.12 into 4.5.10, some simple algebra will produce the form

$$u \equiv x - z = \frac{\mu\chi(z)}{\lambda+2\mu} - \frac{\lambda+\mu}{\lambda+2\mu}\,\overline{\chi}'(z)z + K(\overline{z}) \qquad 4.5.13$$

where $\int \eta(z)dz = \chi(z)$ has been written; the displacement u is also introduced. Employing 4.4.3, the corresponding complex stress potential h may be simply recovered as

$$\frac{h}{2\mu} = \frac{\lambda+\mu}{\lambda+2\mu}\left(\chi(z) + z\overline{\chi}'(z)\right) - K(\overline{z}) \;. \qquad 4.5.14$$

The classical results may be obtained from the book by **MUSKHELISHVILI** (1963) and a summary of his technique will be found in Appendix A2. The results are

$$h_L = \phi(z) + z\overline{\phi'(z)} + \overline{\psi}(z)$$

and 4.5.15

$$u_L = N\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi}(z)$$

where the 'L' denotes a linear quantity. Comparing these with 4.5.13 and 4.5.14, it may be concluded that

$$K(\overline{z}) = -\overline{\psi}(z)$$

and

$$\chi(z) = \frac{\lambda+2\mu}{\lambda+\mu}\,\phi(z)$$

with

$$N = \mu/(\lambda+\mu)$$

are sufficient conditions to permit the two solutions to be identified. Thus it has been shown that the technique employed is consistent with that of **MUSKHELISHVILI** . All that needs to be done now is to verify that the form 4.5.11 is indeed implied by small strains, this follows below.

By employing 4.4.3 and 3.3.6

$$2\mu g'(z) = 2\left(\frac{f'(p)}{p} + \mu\right) \frac{\partial x}{\partial z} \qquad 4.5.16$$

can be obtained. Taking the modulus of this and using 4.2.4(1), gives

$$2\mu|g'(z)| = \left|\frac{f'(p)}{p} + \mu\right| p \quad . \qquad 4.5.17$$

On applying the small strain criteria, as defined by 4.5.1, and expanding $f'(p)$ about the point $p = 2$ to linear terms only, the following results:

$$|g'(z)| = \left|\frac{f'(2)+2\mu}{2\mu} + \frac{f'(2)+\mu E}{2\mu}\right| \qquad 4.5.18$$

where $E = p - 2$. Thus for harmonic materials, it may be ensured that given a sufficiently small strain (as measured by E), that $|g'(z)|$ lies arbitrarily close to a fixed constant. If the material is the semi-linear material or any harmonic material consistent with it for small strains, then the condition $f'(2) = 0$ noted in Section 2.5, forces this constant to be 1. Thus equation 4.5.18 is a statement that $|g'(z)|$ may be arbitrarily close to 1 for small enough strains, and hence the interpretation 4.5.11 is shown to be consistent.

CHAPTER 5 SPECIFIC SOLUTIONS

SECTION 5.1 INTRODUCTION

In this chapter the general solution and techniques developed in Chapter 4 are used to obtain closed form solutions to a variety of problems. Firstly the deformation of annula material configurations is considered, this includes the degenerate case of a cylinder. The boundary conditions are assumed to be constant as a function of the polar angle, thus ensuring that the deformed configuration is again an annulus. Examples of this problem class and their solutions are to be found in Sections 2.6 and 3.4.

Finally, in this chapter the problem of an infinite plate with a circular anomaly is considered. The boundary condition at infinity is taken to be that of a uniaxial tension. The elastic properties of the anomaly will range from being rigid to being simply that of a hole. The solutions of problems of this class are not, in general, of a closed form and integrodifferential equations are involved. The solutions are analysed, as far as is practical, analytically. In the next chapter the solution for a particular material will be continued numerically and pictorial and tabular results will be presented there.

In Section 5.2 the equations of Section 4.4 are restricted such that the boundaries considered are circles centred on the origin. In Section 5.3 the equations of Section 5.2 are used to solve a variety of problems pertaining to annuli. In Section 5.4 the general characteristics of the solutions of Section 5.3 are discussed and common factors are highlighted. In Section 5.5 the solution of three problems concerning an infinite plate containing a circular anomaly are considered. A large proportion of the work in this section is an asymptotic analysis for large $|Z|$.

To conclude Section 5.5, the problem of a finite length crack is formulated, but not solved.

SECTION 5.2 SPECIALISATION TO CIRCULAR BOUNDARIES

In this chapter several problems in which either one or both boundaries are circular are to be considered. For this reason the equations generated at the end of Section 4.4 are presented below, restricted to this form of boundary.

The boundaries are specified as follows:

$$z_1(\bar{z}) = A^2/\bar{z}$$

and 5.2.1

$$\bar{z}_2(z) = B^2/z,$$

representing circles of radii A and B respectively with $A < B$.

It is expedient to define an expression Ψ by

$$\Psi[\lambda,\mu] = \int_{\mu}^{\lambda} \bar{\Phi}^{-1}\left(\pm [g'(\zeta)]^{\frac{1}{2}} [\bar{g}'(B^2/z)]^{\frac{1}{2}}\right)[g'(\zeta)]^{\frac{1}{2}} d\zeta \quad . \qquad 5.2.2$$

The equations to be specialised are 4.4.42 - 45. The process of specialisation is but automatic substitution. The resulting equations are given below:-

$$x_2(z) = x_1(A^2z/B^2) \pm \tfrac{1}{2} [\bar{g}'(B^2/z)]^{-\frac{1}{2}} \Psi[\bar{z}, A^2z/B^2], \qquad 5.2.3$$

$$2\mu x_2(z) = 2\mu g(A^2z/B^2) - h_1(A^2z/B^2) \pm$$
$$\pm \mu [\bar{g}'(B^2/z)]^{-\frac{1}{2}} \Psi[\bar{z}, A^2z/B^2], \qquad 5.2.4$$

$$h_2(z) = 2\mu g(z) - 2\mu x_1(A^2z/B^2) \pm$$
$$\pm \mu [\bar{g}'(B^2/z)]^{-\frac{1}{2}} \Psi[\bar{z}, A^2z/B^2] \qquad 5.2.5$$

and

$$h_2(z) = h_1(A^2z/B^2) + 2\mu \left\{g(z) - g(A^2z/B^2)\right\} \mp$$
$$\mp \mu [\bar{g}'(B^2/z)]^{-\frac{1}{2}} \Psi[\bar{z}, A^2z/B] \qquad 5.2.6$$

respectively. These correspond to the pairs (x,x), (h,x), (x,h) and (h,h). The ordered pair indicates the boundary conditions specified on the circles radius A and radius B; x denotes displacement and h traction boundary conditions. After $g(z)$ has been determined the appropriate solutions are obtained from 4.4.26, 4.4.27, 4.4.37 and 4.4.36 respectively.

The substitution $\bar{z}_1(\bar{z}) = A^2/\bar{z}$ is also required. In practice, however, after 4.4.26 or 4.4.27 or 4.4.36 or 4.4.37 have been employed to determine either x or h, 4.4.3 is then employed to find the other field.

SECTION 5.3 A CLOSED-FORM SOLUTION FOR AN ANNULUS

The configuration adopted is one of an annulus centred on the origin with radii $A < B$. Thus, given the boundary conditions, one of 5.2.3-6 may be employed to determine the unknown function $g(z)$.

The problem to be considered first is that of the radial expansion of the inner boundary whilst the outer boundary remains unstressed. It is supposed that the boundary radius is increased by a factor of α.

The first problem is how to phrase this condition in the required form (4.4.20). As mentioned in Section 5.2 the curve is specified by

$$z = z_1(\bar{z}) = A^2/\bar{z} \ . \qquad 5.3.1$$

Hence the boundary condition is written

$$x_1(z) = \alpha z. \qquad 5.3.2$$

The second problem is the form of the stress boundary condition. As in Section 5.2 the outer boundary is specified by the curve

$$\bar{z} = \bar{z}_2(z) = B^2/z \ . \qquad 5.3.3$$

From equation 4.2.7 the condition of zero traction on the contour 5.3.3 becomes $h_2(z)$ = constant. Without loss of generality this constant may be taken to be zero. The second boundary condition thus becomes

$$h_2(z) = 0 \ . \qquad 5.3.4$$

Now, substituting 5.3.2 and 5.3.4 into 5.2.5, the following may be written

$$0 = 2\mu g(z) - 2\mu\alpha \frac{A^2}{B^2} z \mp \mu \left[\bar{g}'(B^2/z)\right]^{-\frac{1}{2}} \Upsilon\left[z, A^2 z/B^2\right] , \qquad 5.3.5$$

as an equation for $g(z)$, where Υ is defined by equation 5.2.2. An obvious solution of this is

$$g(z) = \eta z \qquad 5.3.6$$

to within an additive constant, where η is a constant depending only on the boundary data.

The question as to whether this solution is unique is at present unanswered. This was discussed at the end of Section 4.4. However, inverting the order in which the boundary conditions are taken yields a similar form for the solution.

Taking the boundary conditions as

$$h_1(z) = 0 \quad \text{on} \quad z_1(\bar{z}) = B^2/\bar{z}$$

and then $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 5.3.7

$$x_2(z) = \alpha z \quad \text{on} \quad \bar{z}_2(z) = A^2/z\ ,$$

and employing 5.2.4, yields

$$2\mu\alpha z = 2\mu g\,(B^2 z/A^2) \pm$$

$$\pm\, \mu[\bar{g}'(A^2/z)]^{-\frac{1}{2}} \int_{B^2z/A^2}^{z} \bar{\Phi}\,(\pm[g'(\zeta)]^{\frac{1}{2}}[\bar{g}'(A^2/z)]^{\frac{1}{2}})[g'(\zeta)]^{\frac{1}{2}}\, d\zeta \qquad 5.3.8$$

as the equation for $g(z)$. The same form of solution is again suggested.

Returning to the problem as originally posed, observe that 5.3.6 with the definition of $\bar{\Phi}(p)$ in 4.4.8, results in

$$\bar{\Phi}(p) = \pm|\eta|\ . \qquad 5.3.9$$

Thus, provided $f'(p) \neq -\mu p$, p is independent of z. Substitution of 5.3.6 and 5.3.9 into 5.3.5 yields the following

$$\pm\,[\,\bar{\Phi}(p) - \tfrac{1}{2}\,p(1-A^2/B^2)]\,\hat{\eta} = \alpha A^2/B^2\ , \qquad 5.3.10$$

where the left-hand side is left in terms of p so that the general properties independent of the particular harmonic material may be discussed. $\hat{\eta}$ is the unit vector in the direction of η. It can be seen immediately that $\hat{\eta}$ is real. From 4.4.7 it is found that

$$\bar{\Phi}(p) - \tfrac{1}{2}p(1-A^2/B^2) \equiv f'(p)/_{2\mu} + \tfrac{1}{2}pA^2/B^2. \qquad 5.3.11$$

This, in view of 4.3.4, is a monotonic increasing function of p, with the value A^2/B^2 when $p = 2$. Equations 4.4.7 and 4.3.3 yield $\bar{\Phi}(2) = 1$ and hence $\eta = 1$, when $p = 2$. Consequently the positive sign in 5.3.9 must be taken and $\eta = \bar{\Phi}(p)$ taken in the neighbourhood of $p = 2$. Equation 5.3.10 then becomes

$$f'(p)/_{2\mu} + \tfrac{1}{2}pA^2/B^2 = \alpha A^2/B^2. \qquad 5.3.12$$

In view of the monotonicity discussed above 5.3.12 shows that p is a monotonic function of α (for fixed A and B). This is also the case for $\bar{\Phi}(p)$ which is clearly now valid for all $p \geqslant 2$. It is worth noting that for $0<\alpha<1$, equation 5.3.12 indicates that $\underline{\Phi}(p)>0$, so that p has a lower limit greater than the critical p_o for this problem. For a given f(p), 5.3.12 may be solved to yield p as a function of α.

The solution for x can now be written from 4.4.26 with $z_1(\bar{z}) = A^2/\bar{z}$, $\eta = \underline{\Phi}(p)$ and 5.3.8, as

$$x = \tfrac{1}{2}p(z - A^2/z) + \alpha\bar{z} . \tag{5.3.13}$$

This solution is now employed to analyse the volume changes taking place in the material, both locally and globally. Using 2.3.18, which gives the local volume ratio J in terms of p and q, with 4.2.4 and 4.2.5, an expression for the volume change may be determined as

$$J = \tfrac{1}{2}p^2 - A^2(\tfrac{1}{2}p - \alpha)^2/R^2, \tag{5.3.14}$$

where $R^2 = z\bar{z}$.

Note firstly that J increases with R. This is in accord with the conclusions reached at the end of Chapter 2. Noting 5.3.12, the dependence of 5.3.14 on α may be eliminated to yield

$$J = \tfrac{1}{2}p^2 - \frac{f'^2(p)}{4\mu^2}\ \frac{B^4}{A^2R^2} \tag{5.3.15(1)}$$

or

$$J = \tfrac{1}{2}(p - \frac{f'(p)}{\mu}\ \frac{B^2}{AR})\ (p + \frac{f'(p)}{\mu}\ \frac{B^2}{AR}). \tag{5.3.15(2)}$$

The requirement of the material impenetrability (1.5.1) may be applied and the following concluded; either

$$\frac{f'(p)}{\mu p} < \frac{AR}{B^2} \quad \text{with} \quad \frac{f'(p)}{\mu p} > -\frac{AR}{B^2} \tag{5.3.16}$$

or

$$\frac{f'(p)}{\mu p} > \frac{AR}{B^2} \quad \text{with} \quad \frac{f'(p)}{\mu p} < -\frac{AR}{B^2} \tag{5.3.17}$$

must hold. The latter (5.3.17) are mutually exclusive, as such 5.3.16 is adopted. The requirement of $f'(p)/\mu p<1$ as in 4.3.10, was noted by JOHN (1960) and KNOWLES & STERNBERG (1975b). It may be seen that 5.3.16(1) is a stronger restriction for this particular deformation.

The condition 5.3.16(2) represents a lower bound on $f'(p)/\mu p$ and thereby contradicts the hypothesis of JOHN (1960) who required $f'(p)/\mu p \to -\infty$ as $p \to 0$. Further, note that 5.3.10 implies that $\bar{\Phi}(p)$ is greater than zero and consequently p must be restricted to be greater than the critical p_o for this problem.

Consider the Jacobian of the transformation evaluated at R = A, J_A. Its dependency on p, or equivalently α, is now discussed. Putting R = A in 5.3.14

$$J_A = \alpha(p-\alpha) \qquad 5.3.18$$

is obtained. Insisting that this is non-negative enables 5.3.16(1) to be recovered with R = A. In Section 4.3 the condition $f'(p) \to \infty$ as $p \to +\infty$ was imposed. As p increases monotonically with α, it is appropriate to impose 5.3.16(1) for all $p>0$, then 4.3.10 follows immediately. Alternatively 5.3.16 may be regarded as limiting the range of α's admissible for this problem. The validity and scope of these restrictions is discussed below.

The dependence of J on p may be investigated using 5.3.15(1) yielding

$$\frac{dJ}{dp} = \frac{p}{2} - \frac{2f'(p)\,f''(p)}{4\mu^2}\quad\frac{B^4}{A^2R^2} \qquad 5.3.19$$

From 4.3.3 the above may be concluded to be greater than zero and hence that:-

J increases initially from unity as α increases from unity, independently of the magnitude of B/A or the form of f(p).

The question as to whether the volume will always increase is not possible to answer for a general material, except that locally at R = A it may be said that

$$J_A \gtreqless 1 \text{ according } \frac{f'(p)}{\mu(p^2-4)^{\frac{1}{2}}} \lesseqgtr \frac{A^2}{B^2}\ . \qquad 5.3.20$$

If $J_A>1$ then $J>1$ for all $R(A \leq R \leq B)$ as J is monotonic increasing in R. However, as may be seen from 5.3.20 whether $J_A>1$ or $J_A<1$ depends intimately on the function f(p). For the semi-linear material the situation is somewhat clearer, in that

$$\frac{f'(p)}{\mu(p^2-4)^{\frac{1}{2}}} = \frac{\lambda+\mu}{\mu}\left(\frac{p-2}{p+2}\right)^{-\frac{1}{2}}$$

and thus

$$\frac{\lambda+\mu}{\mu}\left(\frac{p-2}{p+2}\right)^{\frac{1}{2}} < \frac{A^2}{B^2} \qquad 5.3.21$$

ensures that $J_A > 1$. Consequently, noting that the term dependent on p is monotonic increasing as $p \rightarrow \infty$ from p = 2 and as this term tends to the value 1, 5.3.21 is seen to be a restriction of the validity of this material class. In order that $J_A > 1$ with $\alpha > 1$, p is restricted as

$$2 < p < 2\,\frac{(B^4(\lambda+\mu)^2 + A^4\mu^2)}{(B^4(\lambda+\mu)^2 - A^4\mu^2)} \,. \qquad 5.3.22$$

Having discussed the local volume ratio at R = A, the expression for the ratio of the total deformed to the undeformed volume of the annulus is considered briefly. From 5.3.13 it is possible to write

$$\frac{b^2-a^2}{B^2-A^2} = \frac{p^2}{4} - \frac{1}{4}\,\frac{B(f'(p))^2}{\mu^2A^2} \,,$$

where $a < b$ are the radii of the deformed boundaries. This, on noting 5.3.15 with R = A, is seen to be greater than J_A. Consequently, there is an overall volume increase provided $J_A > 1$. The converse is not necessarily true.

Note that J_A is monotonic decreasing as a function of B/A for fixed α, as should be expected.

It must be borne in mind that the restrictions placed upon f(p) so as to ensure that a reasonable elastic response is predicted, must not be applied unthinkingly. No real material has an infinite elastic regime, and thus there is no reason to impose a reasonable response for the whole range $0 < p < \infty$. Provided the restrictions hold in some domain enclosing the point p = 2, then the theory and material are reasonably applied within that domain. For the semi-linear material 5.3.16(1) cannot be satisfied for all $p > 2$ unless $\lambda < 0$, in which case 5.3.16(2) is invalidated for some p. Indeed, KNOWLES & STERNBERG (1975b) have argued that

$$f''(p)/\mu > 1$$

which, because $f''(2) = \lambda+\mu$ implies that $\lambda > 0$. Thus an inconsistency exists if the restrictions are applied for all p. However, if it is borne in mind that the semi-linear material is a highly restricted form of the constitutive law, being but an extension of the classical linear one, it cannot reasonably be expected to be appropriate as a description of the complete non-linear elastic behaviour of real materials. All the restrictions noted here, in Section 4.3 and in Chapter 2 serve merely to identify the neighbourhood of p = 2 where the description is valid.

For real materials the theory of elasticity fails to be valid beyond some critical range of values of p (containing p = 2), for a number of reasons. For example, the material may rupture or yield at values of p outside this range. Therefore for any given material, provided the inequalities hold within the critical range, the material is deemed to be reasonable. That the material deviates from these conditions outside the range is irrelevant from a physical viewpoint, since elasticity theory is not appropriate there.

Thus, adopting a form of f(p) which fails to satisfy certain inequalities should not mean automatic rejection of that material. It could well be that the elastic domain, where the restrictions hold, is sufficiently large for non-linear characteristics to be discussed.

Consider now the inflation of an annulus. This problem is an interesting variant of that just considered. In some neighbourhood of the undeformed configuration the solutions ought to be formally identical. However, in the problem considered now the possibility of bifurcation into an asymmetrical configuration exists, for some critical value of the pressure.

It is assumed that on the inner boundary $\bar{z}\bar{\bar{z}} = A^2$ there is a uniform hydrostatic pressure σ per unit current area. Employing NANSEN's formula (TRUESDELL & TOUPIN [1960], Eqn.20.8) which relates the deformed to the undeformed normal to a surface, specialised to a curve in plane strain, the expression

$$t = -\sigma\left(N \frac{\partial x}{\partial z} - \bar{N}\frac{\partial x}{\partial \bar{z}}\right) \qquad 5.3.23$$

for the nominal pressure (referred to as T_R in Chapter 2) per unit undeformed length, where N is the unit undeformed normal, is obtained.

For the moment, generalise the equation of the boundary curve to $\bar{z} = \bar{z}_1(z)$ and note that 5.3.23 may be recast as

$$t = -\sigma N \frac{\partial \hat{x}_1}{\partial z} \qquad 5.3.24$$

where $\hat{x}_1(z) = x\left\{z, \bar{z}_1(z)\right\}$ as in Section 4.4. Then, inserting 4.4.32 in 4.4.33 and comparing with 5.3.24 it may be concluded that

$$\frac{d\hat{h}}{dz} = -\sigma \frac{\partial \hat{x}_1}{\partial z}, \qquad 5.3.25$$

where $\hat{h}_1$ is again the restriction of the stress potential field to the contour $\bar{z} = \bar{z}_1(z)$. Integration of 5.3.25 yields

$$\hat{h}_1 = -\sigma\hat{x}_1 \;, \qquad 5.3.26$$

a constant term being ignored. However, from 4.4.3

$$\hat{h}_1 = -2\mu\hat{x}_1 + 2\mu g(z) \qquad 5.3.27$$

and hence

$$\hat{x}_1 = \frac{2\mu g(z)}{2\mu - \sigma} \quad , \quad \hat{h}_1 = \frac{-2\mu\sigma g(z)}{2\mu-\sigma} \quad . \qquad 5.3.28$$

Thus, although for this problem the boundary condition is in neither of the standard forms, it may be expressed in either of these forms by means of 5.3.28. The problem does, however, become that more difficult insofar as the unknown $g(z)$ becomes an integral part of the boundary condition.

Specify zero traction on the other boundary as

$$h_2(z) = 0 \qquad 5.3.29$$

on $\bar{z} = \bar{z}_2(z) = B^2/z$. Either 5.2.5 or 5.2.6 may be employed to determine $g(z)$. Thus, from 5.2.5 with $\hat{h}_1(z)$ given by 5.3.23(1), an expression for $g(z)$ is obtained as

$$\frac{2\mu}{2\mu-\sigma}\, g(A^2z/B^2) - g(z) = \pm\tfrac{1}{2}\left[\bar{z}'(B^2/z)\right]^{\frac{1}{2}} P\left[\bar{z}, A^2z/B\right]. \qquad 5.3.30$$

Again, as in 5.3.8, it may be noted that one solution is as 5.3.6 and may be written $g(z) = \eta^* z$. The * serves to differentiate this from the constant defined in 5.3.6. On substituting this into 5.3.20.

$$\bar{\Phi}(p)/p = \tfrac{1}{2}(1-A^2/B^2)(\sigma-2\mu)/\left[\sigma-2\mu(1-A^2/B^2)\right] \qquad 5.3.31$$

is obtained. An argument presented above has been employed to infer that the upper sign in 5.3.30 must be taken. The solution for the deformation field is obtained from 4.4.37 with $z_1(\bar{z})=A^2/\bar{z}$,

$$x = \tfrac{1}{2}p(z-A^2/\bar{z}) + 2\mu\eta^* A^2/((2\mu-\sigma)\bar{z}), \qquad 5.3.32$$

where $\bar{\Phi}(p) = \eta^*$ and p is obtained from 5.3.31. p is again independent of z. Comparing 5.3.32 with 5.3.13 yields

$$\alpha = 2\mu\eta^* A^2/(2\mu-\sigma), \qquad 5.3.33$$

and the two solutions are identified. Clearly, σ is determined as a function of α. Thus a relationship is determined; that of the measurable strength of the material.

Noting the requirement 5.3.16(1) which ensures that J is greater than zero and using 5.3.31 and 4.4.7, the condition

$$\alpha < 2\mu(1-A^2/B^2)$$

is obtained. This requirement ensures that $|x|$ is a monotonic increasing function of σ, the applied Cauchy pressure. Thus the magnitude of the true applied stress is bounded above. That this limit exists should not be of concern, considering the above discussion concerning harmonic materials in general. However, this result could have been anticipated from other physical considerations, albeit for this material class. From 5.3.31 it may be calculated that

$$\frac{d\sigma}{dp} = \mu A^2B^2(1-A^2/B^2)\left\{p\bar{\Phi}'(p)-\bar{\Phi}(p)\right\}/\alpha \quad ,$$

where α is the radial stretch on R=A. As a consequence of this expression it is seen that the behaviour of σ as a function of p (or equivalent α) depends on the factor

$$\left\{p\bar{\Phi}'(p)-\bar{\Phi}(p)\right\} .$$

This, using 4.4.7, may be simplified to

$$\frac{p^2}{2}\ \frac{d}{dp}\left[\frac{f'(p)}{\mu p}\right]. \qquad 5.3.34$$

Now, JOHN (1960), KNOWLES & STERNBERG (1975b) and Section 2.5, in requiring a physically reasonable response for hydrostatic stress, infer that $f'(p)/\mu p$ has to be monotonic increasing. Hence given that p is positive, 5.3.32 and the two preceeding expressions are greater than zero. Thus a reasonable, physically based requirement is that σ is a monotonic function of α. Equation 5.3.32 thus limits the range of applicability of the material class. It may be noted that from $\sigma < 2\mu(1-A^2/B^2)$, that this limitation increases in severity as the annulus approaches a cylindrical membrane.

For rubber-like solids it is well known that as the volume enclosed by a circular cylindrical membrane is increased, the inflation pressure rapidly attains a maximum, falls to a minimum and then increases monotonically up to rupture (ALEXANDER [1971]). However, the mode of deformation between the maximum and minimum is asymmetrical in nature. It may be noted that the solution corresponding to the form 5.3.6 and 5.3.32 is symmetric. Thus experimental evidence predicts an upper bound to the applied pressure, such that the deformation is symmetric. It is conceivable that other solutions of 5.3.5 exist such that an asymmetric deformation results. However, it must be emphasised that the constitutive law considered here is not appropriate for rubber-like solids, since the bulk modulus is of the same order as the shear modulus, whereas for rubber-like solids their ratio is of the order of 10^4.

As an illustration that the sign of $\underline{\Phi}(p)$ depends on the boundary data and that this degree of indeterminacy presents no fundamental problems, consider the case of an annulus as before but with a solid interior and a boundary condition specified on R=B of a fixed displacement. The boundary condition on $z\bar{z}=A^2$ is therefore

$$x_1(z) = z, \qquad (1)$$

and that on $z\bar{z}=B^2$ is

$$(2) \qquad 5.3.35$$

where ρ is real and

$$A/_B < \rho . \qquad (3)$$

Substitution of 5.3.35(1), (2) and (3) into 5.2.3 yields

$$(\rho - A^2/_B 2)z = \pm \tfrac{1}{2}[\bar{g}'(B^2/z)]^{-\frac{1}{2}}\, \Psi\, [\bar{z}, A^2 z/_B 2]. \qquad 5.3.36$$

A solution of the form 5.3.6 is again possible. Inserting into 5.3.36 yields

$$\tfrac{1}{2}p\eta/\bar{\underline{\Phi}}(p) = (\rho - A^2/_B 2)/(1 - A^2/_B 2). \qquad 5.3.37$$

As from 5.3.37, η is real 5.3.9 may be written as

$$\bar{\underline{\Phi}}(p) = \pm\eta, \qquad 5.3.38$$

and 5.3.37 becomes

$$\tfrac{1}{2}p = \pm(\rho - A^2/_B 2)/(1 - A^2/_B 2). \qquad 5.3.39$$

The requirement that p be positive for all ρ such that 5.3.5(3) holds, indicates that the positive sign must be taken, at least in some neighbourhood of p = 2. Indeed taking the lower bound of $\rho = A/B$, 5.3.39 gives

$$p > 2A/(A + B) \qquad 5.3.40$$

as a limit.

From 4.4.26, with $z_1(\bar{z}) = A^2/_{\bar{z}}$, the solution for x in the form

$$x = \tfrac{1}{2}p(z - A^2/_{\bar{z}}) + A^2/_{\bar{z}} \qquad 5.3.41$$

is obtained. As previously, using $J = \frac{1}{2}(p^2 - q^2)$ and 4.2.4 with 4.2.5 to evaluate the local volume change on R=A,

$$J_A = p - 1 \qquad 5.3.42$$

results, which again is required to be positive. Thus from 5.3.39, imposing 5.3.42, a requirement of

$$\rho > \tfrac{1}{2}(1 + A^2/_B 2), \quad p > 1$$

is obtained.

Now, recalling the definition of p_o from Section 4.3, it is seen that $\eta = \bar{\underline{\Phi}}(p)$ passes through the value $p = p_o$ as ρ decreases, provided $p_o > 1$. In general, however, whether $p_o < 1$ or > 1 depends on the form $\bar{\underline{\Phi}}(p)$. KNOWLES & STERNBERG (1975**b**) present an argument that $p_o > 1$ for any form of $\bar{\underline{\Phi}}(p)$.

If A=0 and the annulus degenerates to a cylinder the solution 5.3.41 reduces to

$$x = \tfrac{1}{2}pz \equiv \rho z. \qquad 5.3.43$$

The deformation is homogeneous, $J = \rho^2$ is automatically positive and the sign of $\bar{\Phi}(p)$ must change as ρ decreases through the value $\rho_0 = \frac{1}{2}p_0$.

Finally in this Section the problem of the shear of an annulus fixed between two circles of fixed radii is considered. This problem was first considered at the end of Chapter 3 where a different solution method was employed. The formal solutions of Section 2.6 are also of relevance.

The boundary curves are as before and the boundary conditions in the conventional form adopted are

$$x_1(z) = z \quad , \quad x_2(z) = z e^{i\theta} \qquad 5.3.44$$

where θ is the fixed angle of applied shear. Insertion of these into 5.2.3 yields

$$(e^{i\theta} - A^2/B^2)\bar{z} = \pm \tfrac{1}{2}[\bar{g}'(B^2/z)]^{-\frac{1}{2}} P[\bar{z}, A^2\bar{z}/B^2] \qquad 5.3.45$$

for the determination of $g(z)$. Again 5.3.6 is a possible solution. Adopting this solution form, 5.3.45 becomes

$$\tfrac{1}{2}p\zeta/\bar{\Phi}(p) = (e^{i\theta} - A^2/B^2)/(1 - A^2/B^2), \qquad 5.3.46$$

with $\bar{\Phi}(p) = \pm|\zeta|$. Taking the modulus of 5.3.46,

$$\tfrac{1}{2}p = \left(1 + \frac{A^4}{B^4} - \frac{2A^2}{B^2}\cos\theta\right)^{\frac{1}{2}}/(1 - A^2/B^2) \qquad 5.3.47$$

results. Clearly p is a monotonic increasing function of θ over the range $0 \leqslant \theta \leqslant \pi$ and

$$1 \leqslant \tfrac{1}{2}p < (1 + A^2/B^2)/(1 - A^2/B^2). \qquad 5.3.48$$

Since $p \geqslant 2$ the positive sign is appropriate and $\bar{\Phi}(p) = |\zeta| \geqslant 1$. Using 5.3.46 and 4.4.26, the deformation field is obtained as

$$x = (e^{i\theta} - A^2/B^2)(z - A^2/\bar{z})/(1 - A^2/B^2) + A^2/\bar{z} . \qquad 5.3.49$$

Notice that this solution is independent of the form of the function f(p). This is contrary to the conclusion of the previous examples.

As previously, note that the volume ratio J is an increasing function of R for any given value of Θ.

$$J_A = \left(\frac{A^4}{B^4} + 2(1-\frac{A^2}{B^2})\cos\Theta \right)/(1-A^2/B^2)^2, \qquad 5.3.50$$

and is positive for Θ in the range $0 \leq \Theta < \pi$, provided

$$\cos\Theta > \tfrac{1}{2}\,(1+A^2/B^2)\ . \qquad 5.3.51$$

This condition restricts Θ to some range $0 \leq \Theta < \Theta_o$ say, where Θ_o depends on the ratio B/A. The smaller the value of B/A the smaller the value of Θ_o. Additionally, $\Theta_o < \frac{1}{2}\pi$ as this corresponds to a plate, $B/A \rightarrow \infty$.

The condition 5.3.51 represents a limit to the angle through which an annulus can be twisted. For example, if B/A=2 the angle Θ_o is about 50^o and the maximum principal stretch about 2. This limitation can be interpreted in various ways:-

(i) It could be that the form of constitutive law is not appropriate for the representation of the elastic deformation described beyond some critical angle Θ_o.

(ii) It may be that internal buckling is initiated at some critical value of $\Theta < \Theta_o$. In this case the deformation does not have the simple form resulting from $g(z) = \lambda z$, but bifurcates into a more complex mode.

(iii) The value Θ_o lies outside the elastic domain of the material in question.

(iv) The solution $g(z) = \lambda z$ of 5.3.45 may not be unique.

It is worth noting that within the allowable range of Θ, J_A decreases monotonically from 1 as Θ increases from zero. Moreover, for R>A it is easy to show that J=1 where $R^2=AB$ and J>1 for $R^2>AB$. The result $J_A<1$ was noted in Section 3.6.

In the corresponding problem for incompressible materials the solution is necessarily of the form $x = ze^{i\Theta(R)}$. Material circles in particular do not change their radius, whereas for compressible solids as considered above, all material circles apart from 3 change radius. In fact, it is easy to show that a non-trivial solution with $r = R$ is impossible for the harmonic material class; this was done in Section 3.6. Thus it is concluded that, in general, non-homogeneous isochoric deformations are not possible for compressible elastic solids.

SECTION 5.4 GENERAL CHARACTERISTICS AND SPECIAL CASES

Each of the solutions presented in Section 5.3 were characterised by the fact that $g(\mathcal{Z}) = \zeta\mathcal{Z}$ for some ζ. A consequence of this was that p was independent of the spatial variable $\mathcal{Z}$, and dependent only on the boundary data. This was shown to be a necessary consequence of circular symmetry for this class of materials. A characteristic of the problems of Section 5.3, is that the boundary data were linear in $\mathcal{Z}$. Indeed, for the strain energy function 4.3.1, if the boundary data were linear in $\mathcal{Z}$ then a solution with $g(\mathcal{Z}) = \zeta\mathcal{Z}$ and p constant is always possible whatever the boundary geometry, as is evidenced by the general forms 4.4.42, 4.4.43, 4.4.44 and 4.4.45. Whether or not this class of solutions is unique is undetermined. The topic of uniqueness was discussed in Section 4.4. Section 5.3 illustrates that solution of the form p = constant contain many interesting traits. Problems in which p varies as a function of position are considered in Section 5.5.

It is of interest to note that a deformation field of the form

$$x = \zeta\mathcal{Z}(\mathcal{Z}/\bar{\mathcal{Z}})^m + K(\bar{\mathcal{Z}}), \qquad 5.4.1$$

where ζ and m are constants and $K(\bar{\mathcal{Z}})$ arbitrary, is such that p is constant. In this connection it is worth remarking that SENSENIG (1965), using a semi-inverse method, obtained a solution to a problem of the deformation of a sector of a circular annulus. His solution in respect of the semi-linear material may be written

$$x = \frac{(\lambda+\mu)}{(\lambda+2\mu)}\,\frac{\mathcal{Z}}{(1+m)}\,(\mathcal{Z}/\bar{\mathcal{Z}})^m + C_1\mathcal{Z}^{1+2m} + C_2\bar{\mathcal{Z}}^{-(1+2m)} \qquad 5.4.2$$

with $\frac{1}{2}<m<0$ and C_1 and C_2 constant. This solution is the only non-trivial solution for compressible elastic solids to be found in the literature on finite strain deformations. For a complete circular annulus continuity arguments require m=0 and 5.4.2 reduces to 5.3.13 with p again constant. This result and those of Section 2.6 would appear to lend weight to an argument that the solution $g(\mathcal{Z}) = \zeta\mathcal{Z}$ is indeed unique for this class of problem where circular

Consider now the boundary value problem of a circular anomaly in an infinite plate. The results of Section 5.3 are applicable in the limit as $B/_A \rightarrow \infty$ provided the boundary condition has meaning there.

Firstly suppose that there is no stress at infinity; that is $h \rightarrow o$ as $/z/ \rightarrow \infty$. On the boundary $z\bar{z}=A^2$ the data is specified in the standard form 4.4.35, the stress condition $h_1(z)$ for the moment being left quite general. The solution of 5.2.6 with $B/_A \rightarrow \infty$, is applicable and reduces to

$$o = h_1(A^2z/_B2)+2\mu\left\{g(z)-g(A^2z/_B2)\right\} \mp \mp \mu\overline{/g'(B^2/z)/}^{-\frac{1}{2}} \mathcal{P}/\bar{z},A^2z/B^2/ , \qquad 5.4.3$$

where $\mathcal{P}/\bar{\lambda},\mu/$ is defined by 5.2.2.

Prior to taking the limit $B/A \rightarrow \infty$, consider a solution of the form $g(z)=\zeta z$. It is of interest to investigate which forms of $h_1(z)$ are admissible with this form. Equation 5.4.3 reduces to

$$o = h_1(A^2z/_B2) + 2\mu\left\{\zeta z-\bar{\zeta}\,\frac{A^2z}{B^2}\right\} \mp \mu pz\,/\overline{1-A^2/_B2}/\, e^{i\omega}. \qquad 5.4.4$$

This on taking the limit $A^2/_B2 \rightarrow o$ becomes

$$h_1(0) + (2/\zeta/\mp p)\mu z e^{i\omega} = 0, \qquad 5.4.5$$

where $\zeta = /\zeta/e^{i\omega}$. This is really an identity for all z and hence

$$h_1(0) = 0$$

and $\qquad 5.4.6$

$$2/\zeta/ \mp p = 0$$

result. These follow from the fact that p is constant given the form $g(z)=\zeta z$. The form $g(z)=\zeta z$ is a solution for any boundary data of stress because 5.4.6 is not a real restriction. The corresponding solutions are obtained from equations 4.4.36 and 4.4.37 as

$$x = \zeta z - h_1(A^2/\bar{z})/2\mu \qquad 5.4.7$$

and

$$h = h_1(A^2/\bar{z}), \qquad 5.4.8$$

where 5.4.6 has been employed.

It will be noticed that 5.4.6(2) is independent of the boundary data $h_1(z)$. Consequently, if a loading path from $h_1(z) = 0$ when $p = 2$ is considered, then these values will not alter as the surface $z\bar{z}=A^2$ is loaded. Thus from 5.4.6(2) with $p = 2$ it may be concluded that $|\eta| = 1$ and

$$\eta = e^{i\omega} \qquad 5.4.9$$

may be written. Additionally it may be noted that 5.4.6 indicates that the negative sign must be adopted in 5.4.4 and 5.4.5 as $p>0$. The value of ω depends on $h_1(z)$ as will be demonstrated.

The resultant force on a curve C, the contour $z\bar{z} = A^2$, in the undeformed configuration is given by the integral of the nominal traction around that curve. Using 4.4.33 this may be written as

$$\oint_c t\,dl = -i[\bar{h}_1(z)]_c \quad , \qquad 5.4.10$$

where dl is the elemental length on C and $[\ ]_c$ denotes the variation on describing C of the enclosed function. As a result of 5.4.10 the condition of zero resultant on C is elegantly expressed as a requirement that $h_1(z)$ is simple valued. The resultant couple is given by

$$\int_c (x_1t_2-x_2t_1)dl = -\tfrac{1}{2}\int_c (\bar{x}dh+xd\bar{h}) \ . \qquad 5.4.11$$

This couple must also vanish on $z\bar{z} = A^2$ which is denoted by C. Thus using 5.4.7 and 5.4.8 with the right-hand side of 5.4.11 and equating, some manipulation yields

$$\int_c \left\{\eta\bar{h}_1(A^2/z) - \bar{\eta}A^2h_1(z)/z^2\right\} dz -$$
$$- \tfrac{1}{2}\eta\, [\bar{h}_1(z)\bar{h}_1(A^2/z)]_c = 0 \quad , \qquad 5.4.12$$

But on C, $\bar{h}_1(A^2/z) = \overline{h_1(z)}$ and therefore as $h_1(z)$ is single valued, the second term is zero. Hence the condition of a zero couple becomes

$$\int_C \left\{ \bar{\zeta}\,\bar{h}_1(A^2/z) - \zeta A^2 h_1(z)/z^2 \right\} dz = 0. \qquad 5.4.13$$

This may be rearranged and 5.4.9 employed to give an expression from which ω may be determined.

$$e^{2i\omega} = A^2 \oint_C z^{-2} h_1(z)\,dz \Big/ \oint_C \bar{h}_1(A^2/z)\,dz. \qquad 5.4.14$$

Thus, as stated, ω is determined from $h_1(z)$ in contrast to the modulus of ζ.

In the pressure loading case (5.3.23) considered, 5.3.28 may be used with $g(z) = \zeta z$ to produce

$$h_1(z) = -2\mu\sigma\zeta z/(2\mu-\sigma) \qquad 5.4.15$$

and 5.4.14 is automatically satisfied, the reason for this is that in the particular case of radial symmetry the angle ω is indeterminate but constant.

If $h_1(z) = Tz$ is taken where $T \equiv N + iS$ is constant, N and S respectively being the normal and shear components of load on $z\bar{z}=A^2$, then from 5.4.16, $\tan\omega = S/N$. When $S = 0$ the solution 5.4.7 with 5.4.8 becomes

$$x = z + TA^2/2\mu\bar{z}$$

and 5.4.16

$$h = TA^2/\bar{z}\ .$$

If $\bar{z}$ is identified with $\bar{x}$ this solution is seen to be the one obtained from the classical linear analysis. The reason **why** this solution **can be** recovered is that $p = 2$ for this problem. The situation is somewhat different in the second problem considered.

Suppose now that zero tractions are applied to the surface $z\bar{z}=A^2$ so that $h_1(z) = 0$. Suppose also that a uniform field at infinity is applied,

$$h_2(z) \to Tz\ , \qquad 5.4.17$$

as $|z| \to \infty$. A more general form of this limit condition is discussed in Section 5.5. Using 5.2.6 as in the previous example

$$T - \mu e^{i\omega}(2/\zeta/ \pm p) = 0 \qquad 5.4.18$$

is obtained, where $\zeta/_{/\zeta/} = e^{i\omega}$. From the case when T = 0 it can easily be deduced that the negative sign is the correct one to be adopted, at least in some neighbourhood of p = 2. From 5.4.18 and using 4.4.7, it may be inferred that

$$\text{(i)} \quad \zeta/_{/\zeta/} = T/_{/T/} \qquad 5.4.19$$

and

$$\text{(ii)} \quad f'(p) = /T/. \qquad 5.4.20$$

From 4.4.36 and 4.4.37 with 5.4.18, 5.4.19 and 5.4.20 the solution pair

$$x = \zeta z - \frac{T}{2\mu}(z - A^2/\bar{z}) \qquad 5.4.21$$

and

$$h = Tz - TA^2/\bar{z} \qquad 5.4.22$$

is obtained. When T is real it is possible to employ 5.4.18 and 4.4.7 to eliminate $/\zeta/$, given that $/\zeta/ = \underline{\Phi}(p)$, to show that

$$2\mu(x-z) = \mu(p-2)z + TA^2/\bar{z}\ . \qquad 5.4.23$$

This, for the semi-linear material, specialises to

$$2\mu(x-z) = \frac{\mu Tz}{\lambda+\mu} + TA^2/\bar{z}\ , \qquad 5.4.24$$

upon using 5.4.20. This has the same form as the corresponding solution in the infinitesimal theory, z and x being identified on the right-hand side. Details of the infinitesimal solution may be found in **MUSKHELISHVILI** (1963, Chap.9), a summary of which may be found in Appendix A2.

In this case the linear solution was only recovered on specialisation to the semi-linear material. Results 5.4.19, 5.4.20 and 5.4.22 illustrate how the solution depends upon the boundary data through the form f(p). This is in contrast to the example considered previously.

SECTION 5.5 PROBLEMS CONCERNING AN INFINITE PLATE

In this section three problems concerning a circular anomaly in an infinite plate are considered. The approach used is that of Section 5.3 and some results are taken from Section 5.4. In this section it will be shown that $g(z)$ being linear is not a restriction of the method. As a consequence of non-linear $g(z)$, p is shown to have a non-zero spatial variation. The method of Chapter 3 is inappropriate for this class of problem. A large proportion of the work presented here is an asymptotic analysis for large $|z|$.

The problem of a circular hole in an infinite plane with zero stresses at infinity and a prescribed stress on $z\bar{z}=A^2$ has been discussed in Section 5.4. The problem of a stress free circular hole with a uniform stress boundary condition at infinity has also been discussed here. In this latter problem h is linear in z and independent of $\bar{z}$ as $|z|\to\infty$. More generally, if the nominal stress is uniform at infinity h may be limited as

$$h\sim\tfrac{1}{2}\left(T_1 + T_2 + i(S_1-S_2)\right)z + \tfrac{1}{2}\left(T_2 - T_1 + i(S_1+S_2)\right)\bar{z} \qquad 5.5.1$$

as $|z|\to\infty$, where T_1, T_2, S_1 and S_2 are constants. Physically T_1 and T_2 are the nominal components of stress at infinity in the X_1 and X_2 directions respectively, whilst S_1 and S_2 are the corresponding shear components.

Henceforth, in using the notation '$\sim$', explicit reference to '$|z|\to\infty$' will be omitted but this is always implied. When $T_2 = T_1 = T$ and $S_1 = -S_2 = S$, the solution is as presented previously. The dependence on $\bar{z}$ in 5.5.1 has not been considered previously. An interesting problem which typifies this dependence is that of a uni-axial tension applied at infinity. In this case $T_1 = T$ and $T_2 = S_2 = S_1 = 0$. The boundary condition 5.5.1 becomes

$$h\sim\tfrac{1}{2}T(z - \bar{z}). \qquad 5.5.2$$

This boundary condition is now considered in detail for two problems; firstly, for a circular hole with zero tractions on $z\bar{z} = A^2$ and secondly, for a circular rigid inclusion of radius A.

Firstly then, the problem of a circular hole under uniaxial tension at infinity. From 4.4.36 with $z_1(\bar{z}) = A^2/\bar{z}$ and $h_1 = 0$ the boundary condition on $z\bar{z} = A^2$ is satisfied by

$$\frac{h}{2\mu} = g(z) - g(A^2/\bar{z}) \mp \tfrac{1}{2} \left[\bar{g}'(\bar{z})\right]^{-\frac{1}{2}} I\left[\bar{z}, A^2/\bar{z}\right] , \qquad 5.5.3$$

where $I[\ ,\ ]$ is defined by 4.4.19. From 5.5.2 a requirement of

$$\frac{\partial h}{\partial z} \sim \tfrac{1}{2}T , \qquad 5.5.4$$

is identified. Hence from 5.5.3 that

$$g'(z) \mp \bar{\Phi}^{-1} (\pm|g'(z)|)\, g'(z)/|g'(z)| \sim \frac{T}{4\mu} . \qquad 5.5.5$$

This expression may be simplified on using 4.4.7 and 4.4.8 to become

$$f'(p)g'(z)/|g'(z)| \sim \tfrac{1}{2}T . \qquad 5.5.6$$

It follows that $g'(z)$ is real as $|z| \to \infty$ and more generally, $g'(z)$ has the same argument as T. Further, it must be positive because $\bar{\Phi}(2) = 1$ and both $\bar{\Phi}(p)$ and $f'(p)$ are monotonic increasing functions of p. Hence, also

$$f'(p) \sim \tfrac{1}{2}T ; \qquad 5.5.7$$

again more generally $f'(p) \sim \tfrac{1}{2}|T|$. It may be deduced that the upper sign in 5.5.3 and 5.5.5 must be taken. Additionally

$$\bar{\Phi}(p) \sim \alpha , \qquad 5.5.8$$

where α is a real constant such that

$$g'(z) \sim \alpha . \qquad 5.5.9$$

Moreover from 5.5.7 and 5.5.8 with 4.4.7, α can be obtained as a function of T as

$$\alpha(T) = \bar{\Phi}\left\{(f')^{-1}(\tfrac{1}{2}T)\right\} = \frac{1}{4\mu} T + \tfrac{1}{2}(f')^{-1}(\tfrac{1}{2}T). \qquad 5.5.10$$

This is monotonic increasing as a function of T with $\alpha(0) = 1$. For the semi-linear material $f'(p) = (\lambda+\mu)/(p - 2)$, and 5.5.10 becomes

$$\alpha(T) = 1 + T(\lambda+\mu)/(4\mu(\lambda+2\mu)) . \qquad 5.5.11$$

Now in view of 5.5.9, $g'(z)$ may be expanded as

$$g'(z) = \alpha + \sum_{n=1}^{\infty} a_n z^{-n} , \qquad 5.5.12$$

where the a_n's are constant. This is the conventional Laurent expansion and is valid for $|z|>R^*$, say. 5.5.9 ensures that no further assumptions are made in adopting this expansion. The first two terms in 5.5.3 may thus be expanded as

$$g(z) - g(A^2/\bar{z}) = \alpha(z - A^2/\bar{z}) + a_1 \ln(z\bar{z}/A^2) + \\ + \sum_{n=2}^{\infty} a_n \,(z^{-n} - \bar{z}^n/A^n). \qquad 5.5.13$$

The expression 5.5.3 then becomes

$$\frac{h}{2\mu} = \alpha(z - A^2/\bar{z}) + a_1 \ln(z\bar{z}/A^2) + \sum_{n=2}^{\infty} \frac{a_n}{A^n} (z^{-n}A^n - \bar{z}^n) - \\ - \tfrac{1}{2} [\bar{g}'(\bar{z})]^{-\frac{1}{2}} \, I \, [\bar{z}, A^2/\bar{z}] . \qquad 5.5.14$$

For the semi-linear material the asymptotic form of 5.5.14 can be solved to yield

$$g'(z) = \alpha + a_2 z^{-2} . \qquad 5.5.15$$

The selection of this form for a general harmonic material is motivated as follows. A strong argument is that a form such as 5.5.15 fully determines a boundary condition of the form 5.5.2. Any further terms in 5.5.15 would have undetermined coefficients. These coefficients, being independent of the boundary data, would correspond to residual stress and deformation fields. An assumption of a natural reference configuration forces these terms to be zero. In view of 5.5.2 it may be argued that $a_n = 0$ for $n \geqslant 3$ since no term involving $\bar{z}^n$ $(n>2)$ can arise from the integral (as shall be demonstrated shortly) to remove contributions of this form from 5.5.13 in the limit as $|z| \rightarrow \infty$. Additionally, in the linear theory there is no logarithmic contribution to x and h and so $a_1 = 0$, as no term may result from the integral in 5.5.14 to compensate for $a_1 \ln(z\bar{z}/A^2)$ in the limit. Lastly, note that in the same way, for small strains, any strain energy density function must approach the semi-linear and so must their solutions similarly converge.

Equation 5.5.14, assuming 5.5.15, reduces to

$$\frac{h}{2\mu} = \alpha(\bar{z}-A^2/\bar{z}) + \frac{a_2}{A^2}(\bar{z}-A^2/\bar{z}) - \tfrac{1}{2}[g'(\bar{z})]^{-\frac{1}{2}} I[\bar{z}, A^2/\bar{z}]. \qquad 5.5.16$$

As α is known from 5.5.11 it remains to determine a_2 by considering the dependence of h on $\bar{z}$ as $|\bar{z}| \to 0$, bearing in mind 5.5.2.

Firstly, from 5.5.16 and 5.5.15

$$\frac{h}{2\mu} \sim \alpha\bar{z} + a_2\frac{\bar{z}}{A^2} - \tfrac{1}{2}\alpha^{-\frac{1}{2}}\int_{A^2/\bar{z}}^{\bar{z}} \bar{\Phi}^{-1}[[g'(\zeta)]^{-\frac{1}{2}}\alpha^{\frac{1}{2}}][g'(\zeta)]^{\frac{1}{2}}\,d\zeta, \qquad 5.5.17$$

where 4.4.19 has been employed and the asymptotic **form of the integral** has yet to be determined.

For large $|\zeta|$, $g'(\zeta) \sim \alpha$ and the integrand in 5.5.18 becomes $\bar{\Phi}^{-1}[\alpha]\alpha^{\frac{1}{2}}$ and the last term in 5.5.18 becomes

$$-\tfrac{1}{2}\bar{\Phi}^{-1}(\alpha)\bar{z} \qquad 5.5.19^{+}$$

for the upper limit only. It remains now to consider the asymptotic form at the lower limit when ζ is small and

$$g'(\zeta) \sim a_2\zeta^{-2}. \qquad 5.5.20$$

The integral becomes

$$\sim \bar{\Phi}^{-1}(\alpha^{\frac{1}{2}}a_2^{\frac{1}{2}}\zeta^{-1})a_2^{\frac{1}{2}}\zeta^{-1}. \qquad 5.5.21$$

As ζ becomes small the argument of $\bar{\Phi}^{-1}$ becomes large, hence the consideration of the behaviour of $\bar{\Phi}^{-1}$ for large values of its argument is motivated.

Now $\bar{\Phi}(p)$ is monotonic increasing and by assumption $\bar{\Phi}(p) \to \infty$ as $p \to \infty$. According to an assumption of KNOWLES & STERNBERG (1975**b**) $\bar{\Phi}(p) \sim p$ as $p \to \infty$ and hence $\bar{\Phi}(p) < p$ for $p < \infty$. However, when considering the problem of the deformation of an annulus, a more stringent requirement was noted, namely

$$\bar{\Phi}(p) \sim \tfrac{1}{2}p(1+A^2/B^2), \qquad 5.5.22$$

which is 4.3.16, with B = A and 4.4.7, has been used. In general then it is appropriate to assume

+**Equation 5.5.18 has not been used.**

$$\bar{\Phi}(p) \sim \nu p \qquad 5.5.23$$

as $p \to \infty$, where ν depends on $\bar{\Phi}(p)$. In light of 5.5.22 $\nu < 1$ is taken. Now 5.5.23 may be equivalently written as

$$\bar{\Phi}^{-1}(p) \sim p/\nu \qquad 5.5.24$$

as $p \to \infty$. Thus 5.5.21 may be written as

$$\sim \frac{\alpha^{\frac{1}{2}} a_2}{\nu} \, \zeta^{-2} \qquad 5.5.25$$

as $/\zeta/^{-1} \to \infty$. The lower limit of the integral in 5.5.17 therefore contributes

$$\frac{-\frac{1}{2} a_2 \bar{z}}{\nu A^2} \quad . \qquad 5.5.26$$

Thus using 5.5.19 and 5.5.26 it is possible to re-write 5.5.19 as

$$\frac{h}{2\mu} \sim \left(\alpha - \tfrac{1}{2} \bar{\Phi}^{-1}(\alpha) \right) z + \left(1 - \frac{1}{2\nu}\right) a_2 \bar{z}/A^2 \,, \qquad 5.5.27$$

hence, it may be seen that the boundary data of the form 5.5.2 is fully determined by a $g'(z)$ of the form 5.5.15.

For the semi-linear material, 5.5.27 becomes

$$\frac{\lambda+2\mu}{\lambda+\mu} \, \frac{h}{2\mu} \sim (\alpha - 1) z + a_2 \bar{z}/A^2 \qquad 5.5.28$$

as $\nu = (\lambda+2\mu)/2\mu$, and hence using 5.5.2 and 5.5.11, a_2 is determined as

$$a_2 = -A^2 T(\lambda+2\mu)/4\mu(\lambda+\mu) \quad . \qquad 5.5.29$$

As mentioned above, this result may be obtained by integration of 5.5.14. Equation 5.5.14 becomes

$$\frac{h}{2\mu} \, \frac{(\lambda+2\mu)}{(\lambda+\mu)} = \alpha \left(z - A^2/\bar{z}\right) + \frac{a_2}{A^2}\left(\bar{z} - A^2/z\right) -$$

$$- \left(\alpha + a_2 \bar{z}^{-2}\right)^{-\frac{1}{2}} \int_{A^2/\bar{z}}^{z} \left(\alpha + a_2 \zeta^{-2}\right)^{\frac{1}{2}} d\zeta . \qquad 5.5.30$$

The integral may be expanded as

$$\int_{A^2/\bar{z}}^{z} \left(\alpha + a_2 \zeta^{-2}\right)^{\frac{1}{2}} d\zeta = \left(\alpha z^2 + a_2\right)^{\frac{1}{2}} - \left(\alpha \frac{A^4}{\bar{z}^2}\right)^{\frac{1}{2}} \quad +$$

$$+\tfrac{1}{2}a_2^{\frac{1}{2}}\ln\left[\frac{(\alpha z^2+a_2)^{\frac{1}{2}} - a_2^{\frac{1}{2}}}{(\alpha z^2+a_2)^{\frac{1}{2}} + a_2^{\frac{1}{2}}}\;\frac{(\alpha A^4/\bar{z}^2 + a_2)^{\frac{1}{2}} + a_2^{\frac{1}{2}}}{(\alpha A^4/\bar{z}^2 +a_2)^{\frac{1}{2}} - a_2^{\frac{1}{2}}}\right] \quad . \qquad 5.5.31(i)$$

The dominant term in 5.5.31(i) as $/z/\to\infty$ is simply the expression $\alpha^{\frac{1}{2}}z$. Inserting this into 5.5.30 and taking the limit as $/z/\to\infty$, allows 5.5.28 to be recovered. However, in the asymptotic analysis of 5.5.31(i) a $\ln\bar{z}$ term has been ignored in favour of $\alpha^{\frac{1}{2}}z$ and if this term is included, 5.5.30 becomes

$$\frac{\lambda+2\mu}{\lambda+\mu}\;\frac{h}{2\mu}\sim(\alpha - 1)z + \frac{a_2\bar{z}}{A^2} - \frac{(a_2)^{\frac{1}{2}}}{(\frac{2}{\alpha})}\ln (a_2^{\frac{1}{2}}\,\bar{z}/A^2) \quad . \qquad 5.5.31(ii)$$

In the specialisation to infinitesimal elasticity, in which case $T/\mu \ll 1$ and hence $a_2/A^2 \ll 1$, and when $\alpha = 1 + O(T/\mu)$, linearisation of 5.5.31(i) yields

$$\int_{A^2/\bar{z}}^{Z}(\alpha + a_2\zeta^{-2})^{\frac{1}{2}}\, d\zeta\sim\alpha^{\frac{1}{2}}z + \tfrac{1}{2}\;\frac{a_2\,\bar{z}}{A^2} \quad , \qquad 5.5.32$$

all terms of the order $(T/\mu)^2$ being neglected. Comparison of the asymptotic forms 5.5.31 and 5.5.32 with 5.5.2 yields

$$a_2 = -2A^2T\;\frac{(\lambda+2\mu)}{4\mu(\lambda+\mu)} \quad , \qquad 5.5.33$$

which replaces 5.5.29 in the linear situation. This value of a_2/A^2 leads to the correct solution for the infinitesimal situation, namely

$$h = \tfrac{1}{2}\,T(z-A^2/\bar{z}) - \tfrac{1}{2}T(\bar{z} - A^2/z) - \tfrac{1}{2}T(z -A^2/\bar{z})A^2/\bar{z}^2 \quad .$$

This result is proved in Appendix A2 using the technique of **MUSKHELISHVILI** (1963).

On the other hand, for the semi-linear material the non-linear solution is given by 5.5.30 with 5.5.31. α is given by 5.5.11 and a_2 by 5.5.29. It is important to appreciate that 5.5.33 must be used in the infinitesimal specialisation. The difference arises because of the conflict between the quantities $A^2/\bar{z}^2$ and a_2/A^2 competing as small quantities in the expansion of

$$\ln\left\{(\alpha A^2/\bar{z}^2 + a_2/A^2)^{\frac{1}{2}} - a_2^{\frac{1}{2}}A\right\}$$

It is necessary to carry out the linearisation before considering the asymptotic form. Note, in particular, that linearisation of the form 5.5.31 is invalid. Moreover, such a linearisation would be incompatible with 5.5.2.

Adopting the form 5.5.23 which arguably is reasonable for all harmonic materials, then 5.5.27 with 5.5.2 yields

$$a_2 = \frac{-TA^2}{4\mu}\,\frac{2\nu}{2\nu-1} \quad . \qquad 5.5.34$$

This is made compatible with the linear theory if

$$\nu = (\lambda+2\mu)/(\lambda+3\mu) \qquad 5.5.35$$

is taken. This ν is generally less than 1, as required.

Thus from an extensive asymptotic analysis the coefficient in the expansion of $g(z)$ as 5.5.15 has been determined. The total solution is then determined by using 5.5.10 to obtain $\alpha(T)$, 5.5.27 for a_2. h is given by 5.5.16 and the deformation field is obtained from 4.4.3. Thus in principle the problem is solved for all harmonic materials.

Consider now the case of a rigid circular inclusion under uniform tension at infinity. The solution of this problem closely parallels that presented above except that 4.4.26 is used rather than 4.4.36. As a consequence of this similarity only an outline is presented here.

From 4.4.26 with $z_1(\bar{z}) = A^2/\bar{z}$ and $x_1(z) = z$ the deformation field is given by

$$x = A^2/\bar{z} + \tfrac{1}{2}\,[\bar{g}'(\bar{z})]^{-\frac{1}{2}}\; I\,[\bar{z}, A^2/\bar{z}], \qquad 5.5.36$$

and satisfies the boundary condition on $z\bar{z} = A^2$. Equation 4.4.3 gives

$$\frac{h}{2\mu} = g(z) - x \; . \qquad 5.5.37$$

This equation allows an asymptotic investigation of condition 5.5.2 in terms of the expression 5.5.28. As in the solution above the form

$$g'(z) = \alpha + a_2 z^{-2} \qquad 5.5.38$$

is taken, with a different a_2, however. It is found that

$$\alpha(T) = \frac{1}{4\mu} T - \tfrac{1}{2}(f')^{-1}(\tfrac{1}{2}T) , \qquad 5.5.39$$

but a_2 is now given by

$$a_2 = \frac{A^2 T \nu}{2\mu} , \qquad 5.5.40$$

with ν given by 5.5.35. The solution 5.5.36 reduces to the solution corresponding to an infinitesimal elastic formulation namely,

$$2\mu(x-z) = \frac{T\mu}{2(\lambda+\mu)} (z-A^2/\bar{z}) + \tfrac{1}{2}T(z-A^2/z) -$$

$$- \tfrac{1}{2} T \frac{(\lambda+\mu)}{(\lambda+3\mu)} (z - A^2/\bar{z}) A^2/\bar{z}^2 . \qquad 5.5.41$$

The solution for the non-linear problem is given by 5.5.38 with 5.5.39 and 5.5.40 inserted into 5.5.36. The stress potential is recovered using 5.5.39. In the next chapter the solution of this problem is continued for the semi-linear material. Graphical and tabular results are also presented, as is a discussion of the numerical method.

To conclude this set of three problems the problem of an elastic circular inclusion with a uniaxial tension applied at infinity is considered. As there is no essential difference between this and the two problems previously considered the solution is dealt with briefly.

Denote all quantities pertaining to the matrix ($z\bar{z} \geqslant A^2$) as in the preceding two problems and denote all those variables or parameters pertaining to the inclusion ($z\bar{z} \leqslant A$) **by means of** a suffix *. Take the general solutions 4.4.14 and 4.4.16 and impose the conditions of continuity of x and h across $z\bar{z} = A^2$, thereby eliminating $K(\bar{z})$ and $K_*(\bar{z})$ to obtain:-

For $z\bar{z} \geqslant A^2$

$$x = \tfrac{1}{2}[\bar{g}'(\bar{z})]^{-\frac{1}{2}} I[\bar{z}, A^2/\bar{z}] + [\mu_* g_*(A^2/\bar{z}) - \mu g(A^2/\bar{z})](\mu_*-\mu) \qquad 5.5.42$$

with

$$h/_{2\mu} = g(z) - x , \qquad 5.5.43$$

and, for $z\bar{z} \leqslant A^2$

$$x = \tfrac{1}{2}[\bar{g}'_*(\bar{z})]^{-\frac{1}{2}} I_*[\bar{z}, A^2/\bar{z}] + (\mu_* g_*(A^2/\bar{z}) - \mu g(A^2/\bar{z}))/(\mu_* - \mu) \qquad 5.5.44$$

with

$$h_*/2\mu_* = g_*(z) - x \, . \qquad 5.5.45$$

$I\,[\;,\;]$ and $I_*[\;,\;]$ are obtained from 4.4.19. It now remains to determine $g_*(z)$.

Firstly it is required that $x(0) = 0$ and that h is finite at $z = 0$. Thus considering 5.5.44 and 5.5.45 as $|z| \rightarrow 0$ then

$$g_*(z) = \alpha_* z \qquad 5.5.46$$

results, where α_* is constant.

Secondly, as mentioned above for the other two problems, 5.5.42 and 5.5.43 can be investigated asymptotically as $|z| \rightarrow \infty$ with h given by 5.5.2. Similar arguments as above lead to

$$g'(z) = \alpha + a_2 z^{-2} \qquad 5.5.47$$

with

$$\alpha - \tfrac{1}{2}\Phi^{-1}(\alpha) = T/4\mu \, , \qquad 5.5.48 \; (5.5.49)$$

and

$$a_2 = -\frac{TA^2}{4\mu^2}(\mu_* - \mu)\left(1 + \frac{\mu_* - \mu}{2\mu\nu}\right)^{-1} , \qquad 5.5.50$$

being determined. Inserting 5.5.46 into 5.5.44 and forcing $|z| \rightarrow \infty$ leads to

$$\alpha_*\left[1 - \frac{\tfrac{1}{2}\Phi^{-1}(\alpha_*)}{\alpha_*}\right] = \frac{\mu(\alpha - \alpha_*)}{(\mu_* - \mu)} \qquad 5.5.51$$

as the consequence of finite stress at $z = 0$. This expression allows α_* to be determined in terms of α. The latter is known from 5.5.48.

The solution 5.5.44 then becomes

$$x = \tfrac{1}{2}\Phi_*^{-1}(\alpha_*) z + \frac{\mu}{(\mu_* - \mu)} \frac{a_2 \bar{z}}{A^2} \qquad 5.5.52$$

with

$$\frac{h}{2\mu} = [\alpha_* - \tfrac{1}{2}\Phi_*^{-1}(\alpha_*)] z - \frac{\mu}{\mu_* - \mu} \frac{a_2 \bar{z}}{A^2} \qquad 5.5.53$$

for $z\bar{z} < A^2$. The solution for $z\bar{z} \geqslant A^2$ is given by 5.5.42 and 5.5.43 with 5.5.46 through to 5.5.51. Within the inclusion the deformation is homogeneous. Indeed this arises for any homogeneous boundary conditions at infinity, just as in the linear theory.

To conclude this section the problem of a finite crack in an infinite plane with a uniaxial tension applied remotely perpendicular to the crack is formulated. Consider the crack specified by

$$z = -\bar{z} \ , \quad /z/ \leqslant 1/2 \qquad 5.5.54$$

which is a crack centre the origin of length 1. Adopt the limiting form 5.5.2 as that representing a uniaxial tension of magnitude T. For this problem T is considered to be negative, as were this not the case, the crack surface boundary data would not be that of a free surface. On the crack surface zero applied traction is specified. This is stated as

$$h_1 = 0 \text{ on } z = -\bar{z}, \quad /z/ < 1/2, \qquad 5.5.55$$

on using 4.4.33. A condition of continuity of the deformation field outside the crack but in the crack plane, is required. This is specified as a requirement that the real component of x is zero on the X_2 - axis. This is best written as

$$x_1 + \bar{x}_1 = 0 \text{ on } z = -\bar{z}, \ /z/ > 1/2 \quad , \qquad 5.5.56$$

although an equivalent form is

$$\hat{x}_1(z,-z) = i[\bar{X}(z) + \overline{X(z)}], \ /z/ > 1/2, \qquad 5.5.57$$

for some $X(z)$.

To summarise then, the solution to this problem is given by 4.4.16 and 4.4.14, yielding h and x respectively, where

$$h \sim \frac{T}{2} (z - \bar{z}) \qquad 5.5.58$$

$$h_1 = 0 \quad \text{on } z = -\bar{z} \text{ with } /z/ < 1/2 \qquad 5.5.59$$

$$x_1 + \bar{x}_1 = 0 \text{ on } z = -\bar{z} \text{ with } /z/ > 1/2 \qquad 5.5.60$$

are the boundary conditions.

Analysis of 4.4.16 with 5.5.58 leads again to the conclusion that

$$g(z) = \alpha + \sum_{n>0} a_n z^{-n} \qquad 5.5.61$$

with

$$\alpha - \bar{\alpha} = 0 \qquad 5.5.62$$

and

$$2\alpha \pm \bar{\Phi}^{-1}(\alpha) = T \,. \qquad 5.5.63$$

Further from 5.5.59 with 4.4.16

$$\frac{h}{2\mu} = \mp \tfrac{1}{2}[\bar{g}'(\bar{z})]^{-\frac{1}{2}} \int_{-\bar{z}}^{z} \bar{\Phi}\, [(g'(\zeta))^{\frac{1}{2}}(\bar{g}'(\bar{z}))^{\frac{1}{2}}][g'(\zeta)]\, d\zeta + g(z) - g(-z) \qquad 5.5.64$$

is obtained. However, this is only valid for $|z| < 1/2$. Consequently it is not possible to extend this to infinity to employ 5.5.58. Using 4.4.3

$$\mathrm{Re}(x) = \mathrm{Re}(g(z)) - \frac{1}{2\mu}\,\mathrm{Re}(h) \qquad 5.5.65$$

may be written and thus 5.5.60 becomes

$$\tfrac{1}{2}\mu\,\mathrm{Re}(h) = \mathrm{Re}(g(z)) \,, \qquad 5.5.66$$

on $z = -\bar{z}$, $|z| > 1/2$. It is at present not known how this condition is dealt with. The problem is left at this point. To date the problem of the finite crack in a finite deformation context has not been solved. It is not known whether a solution exists for materials of the harmonic class.

CHAPTER 6 NUMERICAL CONTINUATION OF A SOLUTION

SECTION 6.1 INTRODUCTION

In this chapter the solution of the problem of a rigid circular inclusion in an infinite material plane subjected to a uniaxial tension at infinity is continued numerically. This problem was considered in Section 5.5. The description 2.5.19 which is that of the semi-linear material is adopted. Even for this simple material the solution possesses many interesting characteristics.

In Section 6.2 a new terminology is introduced for two reasons. Firstly, the general solution of Section 4.4 was originally developed using the notation which will now be introduced. It is easier to continue the solution of Section 5.5 in the original notation and to simply point out the equivalences where necessary. Secondly, a major characteristic of the solution is a singularity with an associated indeterminacy which are also better discussed in terms of the original notation.

Section 6.3 contains a short description of the numerical algorithm to be employed. In Section 6.4, the main section of this chapter, the graphical results are presented for various values of the traction applied at infinity. Some results are also presented which correspond to the case where there is an applied shear in addition to the uniaxial tension. This situation corresponds to the T of equation 5.5.2 being complex, and as such the asymptotic analysis of that section ceases to be valid. Thus the asymptotic analysis of Section 5.5 is repeated for a complex T producing results which are at variance to those used when producing the graphical results. A counter-example is employed which throws some doubt on the validity of the results of the asymptotic analysis of this section. Finally, in this section some graphical results are presented which illustrate the existence of a singularity.

The existence of this singularity is further confirmed by the non-convergence of the numerical procedure in some cases, examples of which may be seen in this section. The singularity is discussed analytically and it is seen to result from a branch point of an integrand. This discussion is facilitated by the existence of an analytic representation of the solution which will be obtained for this particular case. Whether the singularity in the solution is inherent in the problem, or is introduced by the method, is a question still to be resolved.

SECTION 6.2 A DIFFERENT TERMINOLOGY

The terminology and notation presented here are different from those employed in Chapters 4 and 5. However, the solution method is essentially the same and there is little difficulty in converting to either of the two representations of the solution. The terminology previously employed has the advantage that the sign of equation 4.4.7 is easier to resolve. In the notation to be introduced here the indeterminacy is lodged elsewhere allowing a change of this sign within the material to take place. A comparative analysis, however, reveals that this change of sign has to be made in order to keep the $\underline{\Phi}(p)$ of equation 4.4.7 of constant sign throughout the material body.

As the solution method in the new notation is not fundamentally different from the one presented in Chapter 4, the results are simply quoted and the corresponding references are provided in brackets. The general solution for harmonic materials may be developed as

$$x = \frac{1}{2}\int^{\mathcal{Z}} p(t,\bar{\mathcal{Z}})\frac{\hat{f}(t)}{|\hat{f}(\mathcal{Z})|}\,dt + \bar{r}(\bar{\mathcal{Z}}) \qquad \begin{matrix}6.2.1\\(4.4.14)\end{matrix}$$

and

$$\frac{h}{\mu} = 4\int^{\mathcal{Z}}\hat{f}(t)dt - 2x$$

$$= \int^{\mathcal{Z}}\hat{f}(t)\left[4 - \frac{p(t,\bar{\mathcal{Z}})}{|\hat{f}(\mathcal{Z})|}\right]dt - 2\bar{r}(\bar{\mathcal{Z}}), \qquad \begin{matrix}6.2.2.\\(4.4.16)\end{matrix}$$

with

$$\left|p + \frac{f'(p)}{\mu}\right| = 4\,|\hat{f}(\mathcal{Z})| \quad , \qquad \begin{matrix}6.2.3\\(4.4.7 \text{ and } 4.4.8)\end{matrix}$$

where $\hat{f}(\mathcal{Z})$ is analytic and $\bar{r}(\bar{\mathcal{Z}})$ anti-analytic. Both $f(\mathcal{Z})$ and $\bar{r}(\mathcal{Z})$ are arbitrary and are determined from the boundary data.

The consequences of the boundary data

$$x = \mathcal{Z} \text{ on } R = a \qquad 6.2.4$$

and

$$h \sim \frac{T}{2}(\mathcal{Z}-\bar{\mathcal{Z}}) \text{ as } R \to \infty \quad , \qquad 6.2.5$$

are now investigated. A general harmonic material description is first used and is then specialised to the semi-linear description as and when necessary. In employing the solution 6.2.1 to 6.2.3, the form 5.5.15 will be seen to be determined absolutely with few assumptions needing to be made. The analysis required in this section differs from that of Section 5.5 only in the order of application and, as such, a summary is simply included here.

Consider equations 6.2.2 and 6.2.5 now. These may be differentiated with respect to z and the limit condition applied to yield

$$\mu \hat{f}(z) \left[4 - \frac{p(z,\bar{z})}{|\hat{f}(z)|} \right] \sim \frac{T}{2} \quad . \tag{6.2.6}$$

Differentiating the same pair of equations with respect to $\bar{z}$ and again applying the limit condition, the following is obtained:-

$$\mu \left(\int^{z} \frac{\hat{f}(t)}{|\hat{f}(z)|} \left[\bar{z} p(t,\bar{z}) \bar{\hat{f}}(z) d \frac{\bar{\hat{f}}(z)}{d\bar{z}} - \frac{\partial p(t,\bar{z})}{\partial \bar{z}} \right] dt - 2\bar{r}'(\bar{z}) \right) \sim -T/2. \tag{6.2.7}$$

Now, 6.2.6 indicates how the integrand of 6.2.2 behaves asymptotically for large $|z|$ and as a consequence of this behaviour it may be deduced that

$$\bar{r}(\bar{z}) \sim \frac{T}{4\mu} \bar{z} \quad . \tag{6.2.8}$$

This result may then be employed in 6.2.7 to yield

$$\int^{z} |f(z)| \frac{\partial}{\partial \bar{z}} \left(\frac{p(z,\bar{z})}{|\hat{f}(z)|} \right) dz \sim 0. \tag{6.2.9}$$

The integral is zero and path independent, hence it can be deduced that the integrand is zero. Thus, using 6.2.3, the result

$$|\hat{f}(z)| \frac{\partial}{\partial \bar{z}} \frac{f'(p)}{|\hat{f}(z)|} \sim 0 \tag{6.2.10}$$

follows. From this it may be concluded that $f'(p)/|\hat{f}(z)|$ is asymptotically independent of $\bar{z}$ for large $|z|$. This is best written as

$$p(z,\bar{z}) \sim r(z)\, |\hat{f}(z)|$$

with (6.2.11)

$$f'\left(\frac{p(z,\bar{z})}{\mu}\right) \sim (4 - r(z))/|\hat{f}(z)| \quad .$$

The last of these is a consequence of equation 6.2.3. The other root, $/f(z)/\sim 0$, of 6.2.10 is discounted as being singular since $T \to 0$ requires that $/\hat{f}(z)/\to 2$ for all z.

From the fact that both p and $/\hat{f}(z)/$ are real and positive it must be concluded from 6.2.11 that $r(z)$ is real and positive. Assuming this over some neighbourhood of infinity it may be concluded that

$$r(z)\sim r^* ,$$

which is a real positive constant. Inserting this result into the asymptotic form 6.2.3 and writing

$$p(z,\bar{z})\sim p^* \geqslant 0$$

and $\qquad$ 6.2.12

$$/\hat{f}(z)/\sim f^* \geqslant 0$$

results in the expression

$$/\mu p^* - f'(p^*)/=4\mu p^*/r^* , \qquad 6.2.13$$

for a specified f(p). Additionally, from the asymptotic form for $/\hat{f}(z)/$, it may be concluded that

$$\hat{f}(z) = a_o + \sum_{n=1}^{\infty} a_n z^{-n} , \qquad 6.2.14$$

and from 6.2.6 that

$$\mu a_o [4-r^*] = T/2 . \qquad 6.2.15$$

Thus this demonstrates that a_o in 6.2.14 is real for all harmonic materials.

At this point in the analysis the semi-linear material description is selected with

$$f(p) = \frac{\lambda+\mu}{2}(p-2)^2 . \qquad 6.2.16$$

Adopting this, 6.2.6 may be employed to relate p*,T and f* as

$$p^* = 4f^* \mp /T//2\mu \qquad 6.2.17$$

which, in 6.2.13, becomes

$$/(4f^* \mp \frac{/T/}{2\mu})(\lambda+\mu) - 2(\lambda+\mu)/ = 4\mu f^* , \qquad 6.2.18$$

when 6.2.11 is invoked. In this expression there are a great many unresolved signs which have to be resolved. The manner in which this is done is tedious as all cases must be considered. Once the signs are resolved, a_o ($f^* = |a_o|$) is determined as a function of T with the branches being selected such that the basic conditions

$$p > 0$$

and

$$\lim_{T \to 0} a_o = \tfrac{1}{2}$$

are not violated. On imposing these conditions the following are obtained

$$a_o = \tfrac{1}{2} + \frac{T}{8\mu}\frac{(\lambda+2\mu)}{(\lambda+\mu)} \qquad 6.2.19$$

$$\text{for } T \geq -4\mu(\lambda+\mu)/(\lambda+2\mu) \qquad 6.2.20$$

with

$$p^* = 2 + T/2(\lambda+\mu) \qquad 6.2.21$$

and

$$r^* = 4\left[1 - \frac{T(\lambda+\mu)}{4\mu(\lambda+\mu) + T(\lambda+2\mu)}\right]. \qquad 6.2.22$$

Now, when considering 6.2.4, the boundary data on the surface of the inclusion, equation 6.2.1 must be restricted to the circle $R = a$. Differentiating the reduced 6.2.1 with respect to the angular variable of integration and applying 6.2.4,

$$\bar{r}'(ae^{i\theta}) = \left(\tfrac{1}{2}p(ae^{i\theta}, ae^{-i\theta})\frac{\hat{f}(ae^{i\theta})}{|\hat{f}(ae^{i\theta})|} - 1\right) e^{2i\theta}, \qquad 6.2.23$$

is obtained. This is valid for all harmonic materials. For the semi-linear material,

$$p = \frac{2}{(\lambda+2\mu)}\left[(\lambda+\mu) + 2\mu/\hat{f}(\bar{z})|\right] \qquad 6.2.24$$

and 6.2.23 may be extended into the region $|\bar{z}| > a$ resulting in

$$\bar{r}(\bar{z}) = a^2\bar{z}^{-1} + \frac{a^2}{\lambda+2\mu}\int \bar{z}^{-2}\left[(\lambda+\mu)\frac{\|\hat{f}(a^2\bar{z}^{-1})\|}{\hat{\bar{f}}(\bar{z})} + 2\mu\hat{f}(a^2\bar{z}^{-1})\right]d\bar{z}. \qquad 6.2.28^{+}$$

In this expression $\|\hat{f}(a^2\bar{z}^{-1})\|$ is not the conventional modulus but it is such that the modulus of $\hat{f}(\bar{z})$ is determined and $a^2/\bar{z}$ is substituted for all $\bar{z}$ terms. For instance, if

+ **Equation numbers 6.2.25 — 27 have not been used.**

$$\hat{f}(z) = a_o + b_o z$$

then

$$\bar{\hat{f}}(z) = \bar{a}_o + \bar{b}_o \bar{z}$$

and

$$//\hat{f}(a^2\bar{z}^{-1})//^2 = a_o{}^2 + a_o\bar{b}_o\bar{z} + \bar{a}_o b_o a^2 \bar{z}^{-2} + b_o a^2. \qquad 6.2.29$$

From 6.2.8 $\bar{r}(\bar{z})$ is asymptotically linear in $\bar{z}$. A trivial analysis using 6.2.29 indicates that the dominant terms of $\hat{f}(z)$ are

$$\hat{f}(z) = a_o + \alpha z^{-2} \qquad 6.2.30$$

in order that $\bar{r}(\bar{z})$ be linear in the limit. The coefficient α is determined using the expressions 6.2.28 and 6.2.29 with the limit 6.2.8. After α has been determined 6.2.30 may be used in 6.2.28 to yield an expression for $\bar{r}(\bar{z})$ as

$$\bar{r}(\bar{z}) = a^2\bar{z}^{-1}\left(1 - \frac{2a_o\mu}{(\lambda+2\mu)}\right) + \frac{\mu}{\lambda+2\mu}\;\frac{2\alpha\bar{z}}{a^2} + a^2\frac{(\lambda+\mu)}{\lambda+2\mu}\;I_2 \qquad 6.2.31$$

where

$$I_2 = \int \frac{(a_o{}^2 + a_o\alpha(\bar{z}^{-2} + \bar{z}^2/a^4) + \alpha^2/a^4)^{\frac{1}{2}} d\bar{z}}{(a_o\bar{z}^2 + \alpha)} \qquad 6.2.32$$

with

$$\alpha = \frac{T(\lambda+2\mu)a^2}{8\mu^2}. \qquad 6.2.33$$

With α as above and a_o as in 6.2.19, it is to be noted that I_2 of 6.2.32 is path independent outside R = a.

With the solution as generated above, the boundary condition of place on the surface of the inclusion and that of traction at infinity are both satisfied. It is of interest to note that there appears to be some measure of non-uniqueness in the form for $\hat{f}(z)$. The form selected ensures that the boundary conditions are satisfied. Additional terms such as

$$a_n z^{-n} \qquad n \geqslant 3$$

are admissible and do not invalidate any of the imposed conditions nor the foregoing analysis. The coefficients a_n are not connected with any boundary conditions and may be arbitrarily assigned. However, if these

a_n are to be constant, each must be zero as each could correspond to a residual stress and deformation field. The a_n could, however, be used to accommodate a body force distribution provided the latter were to admit a scalar potential. If the a_n were considered functions of the boundary data so as not to preclude a natural reference configuration, this would be tantamount to allowing mechanical action at a distance.

To summarise then, a complete solution may be written as

$$x = \bar{r}(\bar{z}) + \frac{2\mu}{(\lambda+2\mu)} \left[\bar{a}_o z - \alpha z^{-1}\right] + \frac{\lambda+\mu}{\lambda+2\mu} I_1 \qquad \text{(i)}$$

and

$$\frac{h}{\mu} = 4\int \hat{f}(z)\,dz - 2x \qquad \text{(ii)}$$

with

$$I_1 = \int \frac{a_o + \alpha z^{-2}}{(a_o^2 + a_o\alpha(\bar{z}^{-2}+z^{-2}) + \alpha^2 z^{-2}\bar{z}^{-2})^{\frac{1}{2}}}\,dz \qquad \text{(iii)}$$

and

$$\bar{r}(\bar{z}) = \frac{a^2}{\bar{z}}\left(1 - \frac{2a_o\mu}{\lambda+2\mu}\right) + \frac{\mu 2\alpha\bar{z}}{(\lambda+2\mu)a^2} + \frac{a^2(\lambda+\mu)}{(\lambda+2\mu)} I_2 \,, \qquad \text{(iv)} \quad 6.2.34$$

where

$$I_2 = \int \frac{(a_o^2 + a_o\alpha(\bar{z}^{-2}+\bar{z}^2/a^4) + \alpha^2/a^4)^{\frac{1}{2}}}{(a_o\bar{z}^2 + \alpha)}\,d\bar{z} \,, \qquad \text{(v)}$$

and

$$a_o = \frac{1}{2} + \frac{T}{8\mu}\,\frac{\lambda+2\mu}{\lambda+\mu} \qquad \text{(vi)}$$

with

$$\alpha = \frac{T(\lambda+2\mu)a^2}{8\mu^2} \,. \qquad \text{(vii)}$$

Another solution form which may be compared with that of Section 5.5 results if it is noted that 6.2.4 is satisfied when

$$\bar{r}(\bar{z}) = \frac{a^2}{\bar{z}} - \frac{1}{2}\int^{a^2/\bar{z}} p(t,\bar{z})\,\frac{\hat{f}(t)^{\frac{1}{2}}}{\hat{\bar{f}}(z)^{\frac{1}{2}}}\,dt \,, \qquad 6.2.35$$

and then

$$x = \frac{a^2}{\bar{z}} + \tfrac{1}{2} \int_{a^2/\bar{z}}^{z} p(t,\bar{z}) \frac{\hat{f}(t)^{\frac{1}{2}}}{\bar{\hat{f}}(\bar{z})^{\frac{1}{2}}} \, dt \qquad 6.2.36$$

with h still being given by 6.2.2. Employing 6.2.24, 6.2.30 and 6.2.34(vi) and (vii) yields

$$x = \frac{a^2}{\bar{z}} + \frac{2\mu}{(\lambda+2\mu)} \left[\bar{a}_o z - a_o \frac{a^2}{\bar{z}} - \alpha z^{-1} + \alpha \frac{\bar{z}}{a^2} \right] + \frac{(\lambda+\mu)}{(\lambda+2\mu)} I_3 \qquad 6.2.37$$

where

$$I_3 = \int_{a^2/\bar{z}}^{z} \frac{(a_o^2 + a_o\alpha(z^{-2}+\bar{z}^{-2}) + \alpha^2 z^{-2}\bar{z}^{-2})^{\frac{1}{2}}}{(a_o + \alpha\bar{z}^{-2})} \, dz \ . \qquad 6.2.38$$

In this form the solution is directly comparable to the one produced in Section 5.5. The two solutions 6.2.34(i) and 6.2.37 are demonstrably equivalent. Additionally, provided the integrals in 6.2.34(i) are taken to be path integrals from a fixed point z^* say, then provided

$$z^*\bar{z}^* = a^2 \qquad 6.2.39$$

they produce the same numerical results. Each of these solutions has been employed in tabulating the deformation field.

SECTION 6.3 NUMERICAL METHOD

In all, three programs have been written to tabulate the deformation field corresponding to the problem of Section 5.5. The first program is employed to integrate equations 6.2.34(i) to (vii). The other two programs are used to integrate 6.2.37 and 6.2.38 with 6.2.34(vi) and (vii), with these two programs differing only in mesh adopted. The reason for this seeming multiplicity of effort is to obtain an understanding of the branch points of the solution functions. The question of singular points and branches is considered in the next section.

In all three programs all lengths are normalised with respect to the inclusion radius as a unit. Values of the material constants $\lambda = 1.0$ and $\mu = 0.5$ are adopted. These values correspond to a Young 's modulus E, of 1.3 and to a Poisson's ratio of 0.3. The applied traction ought to be compared with the former. The applied traction is varied to include a remotely applied shear.

Equations 6.2.34 and 6.2.38 are to be tabulated. Their form is such that an essential routine in any tabulation procedure is that of quadrature. The generic quadrature procedure employed is that of PATTERSON (1968). Essentially, this procedure is a modified Gaussian algorithm:

> Starting with a 3-point Gauss rule a new 7-point rule is derived, 3 of whose abscissae coincide with the original abscissae. The remaining 4 pivots are chosen to give the greatest possible increase in polynominal integrating degree. This process is repeated and further rules with n pivots and of precision m are generated. The sequence is

n = 7	with	m = 11
15		23
31		47
63		95
127		191
255		383

The successive rules are applied in turn until two consecutive results differ by some specified small quantity.

In this application when the quadrature is of a complex nature the real and imaginary parts are separately converged.

The method was selected for two reasons:-

- It minimised the number of function evaluations required for a given accuracy.
- The method did not require evaluation of the integrand at the end points.

This second reason was of particular importance considering the branch points. The quadrature sub-routine used in the procedure DØIACF which is from the NAG (Nottingham Algorithm Group) Library.

SECTION 6.4 GRAPHICAL RESULTS AND DISCUSSION

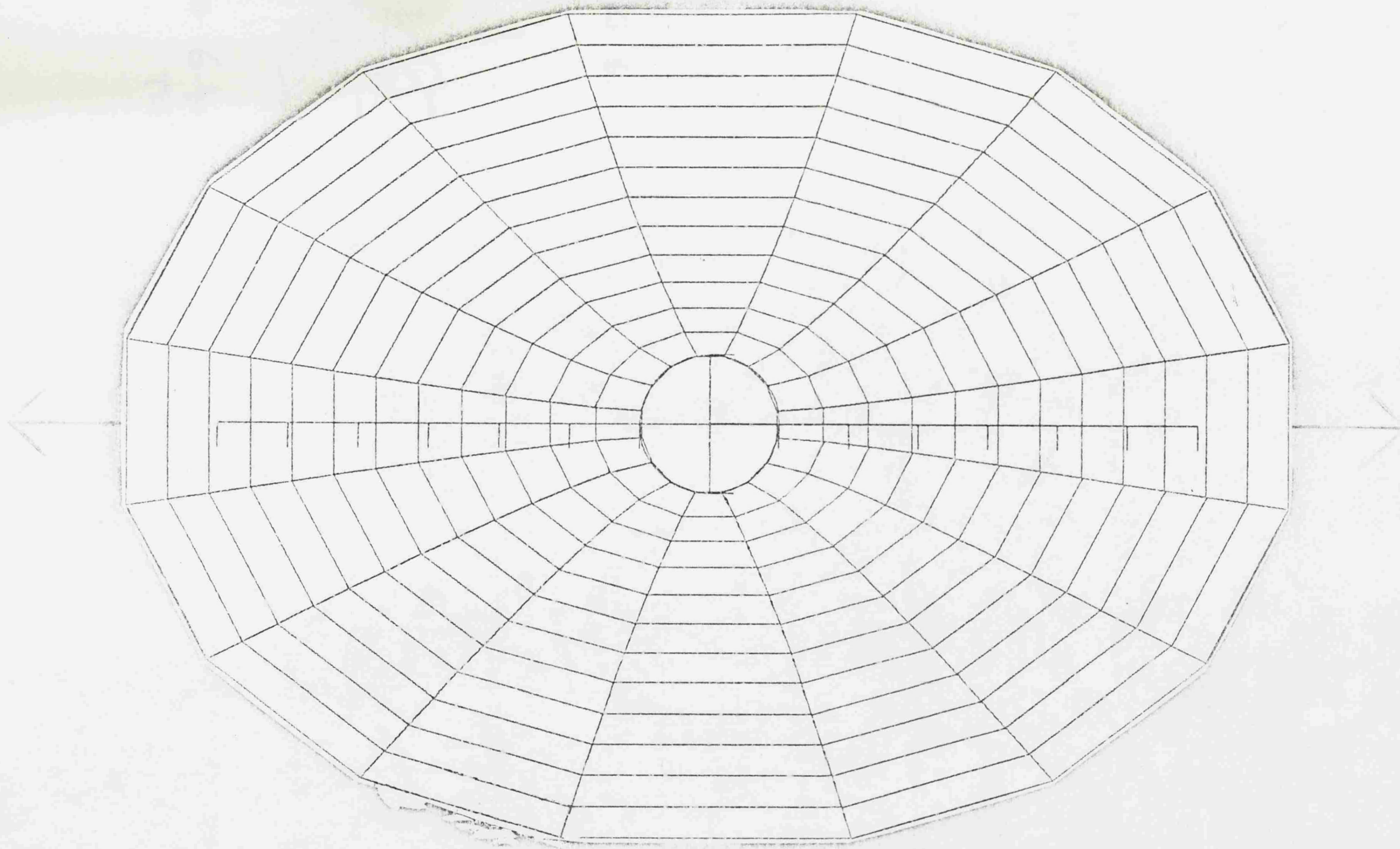
The deformation fields presented graphically in this section are those corresponding to the problem discussed in Section 6.2. Deformation fields corresponding to generalisations of this problem are also presented. The figures presented are those of the deformation of one of two embedded grids:-

(i) CIRCULAR - a set of concentric circles centre the origin. The innermost circle with radius 1 represents the inclusion boundary. The radii of the remaining circles are incremented by 0.5. Superimposed on this set of circles is a set of equally spaced radial lines with an angular separation $\pi/_8$ radians. In general, the first radial line is at $\pi/_{16}$ radians to the real axis; the reason for this will become apparent subsequently.

(ii) RECTANGULAR - a regular square mesh of interstitial distance 0.5. The inclusion radius is taken as 1 to define the unit of length.

The programs are written so as to employ the same scale in both co-ordinate directions. Thus the inclusion maintains a circular profile and the deformation is not masked by co-ordinate distortion. There are only two scales employed and these may be identified by the abscissas markings which are at unit intervals.

Figures 6.4.1 are a series of deformed circular meshes for various values of uniaxial tension applied at infinity. The range of applied tractions is 0.2 to 1.0.



T= 2.

FIG 6.4.1(i) T=2

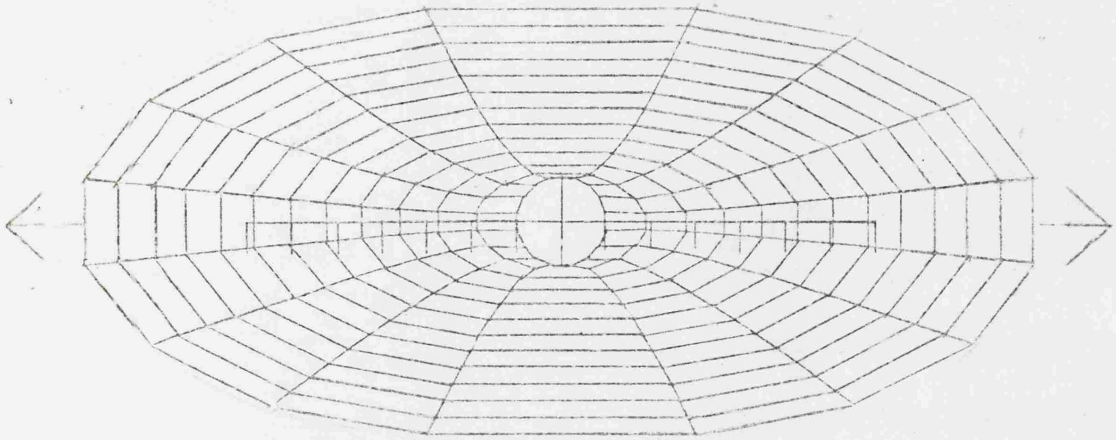


FIG 6.4.1 (iii)
T= .6

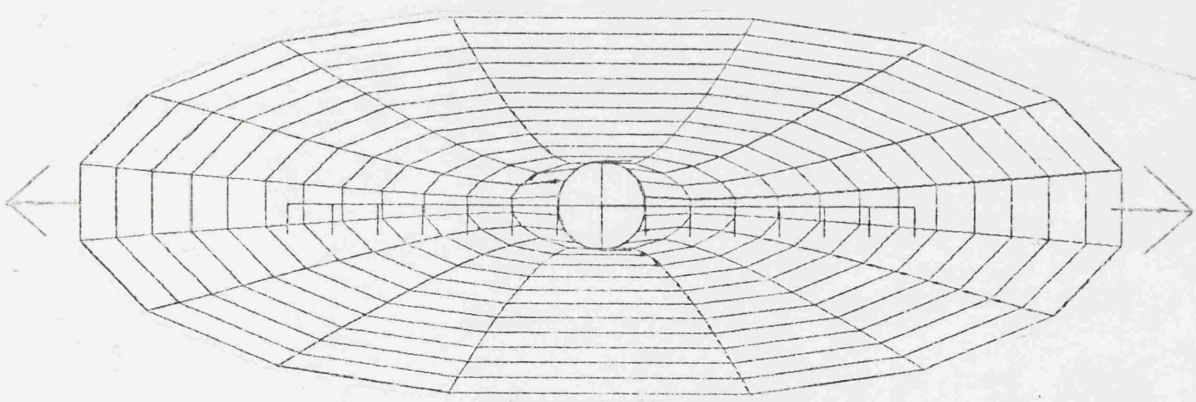


FIG 6.4.1 (iv)
T= .8

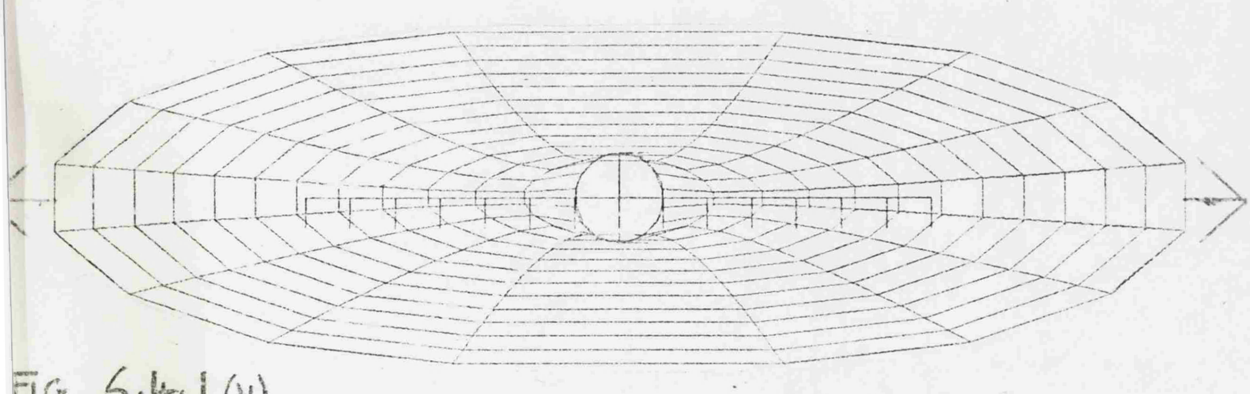
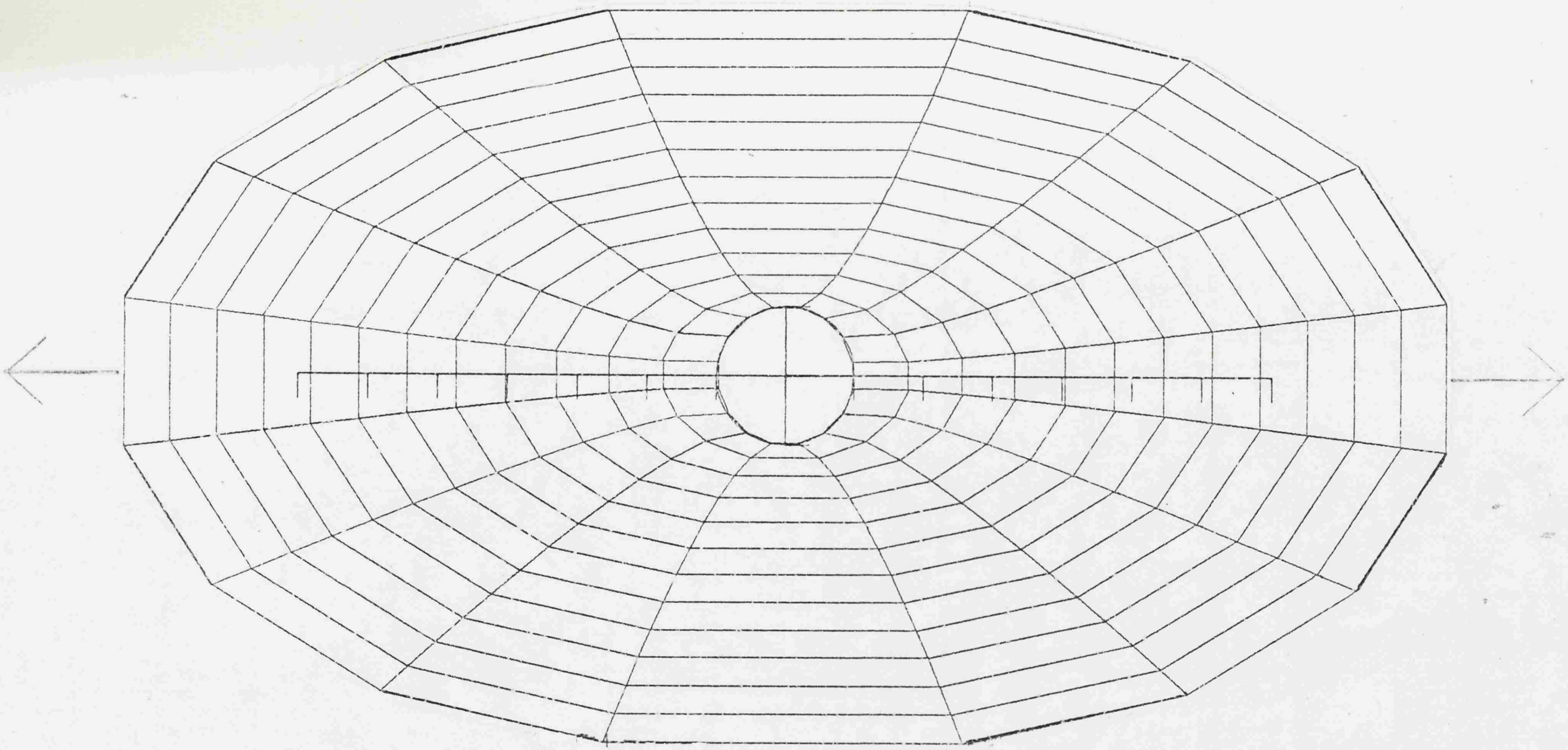


FIG 6.4.1 (v)
T=1.0



T=.4

FIG 6.4.1(ii) T=.4

The second group of results, figures 6.4.2, is again for the same problem of uniaxial tension as considered previously.

The exception to this is the last figure which is included here for completeness and is the first example of a deformation field where a remotely applied shear has been admitted. It also illustrates the singularity which will be discussed extensively towards the end of this section. The first in the series of figures shows the deformation of a complete rectangular grid surrounding the inclusion. The rest of the figures are representations of the region to the right of $X_1 = 1.0$ corresponding to the heavy line in figure 6.4.2(i).

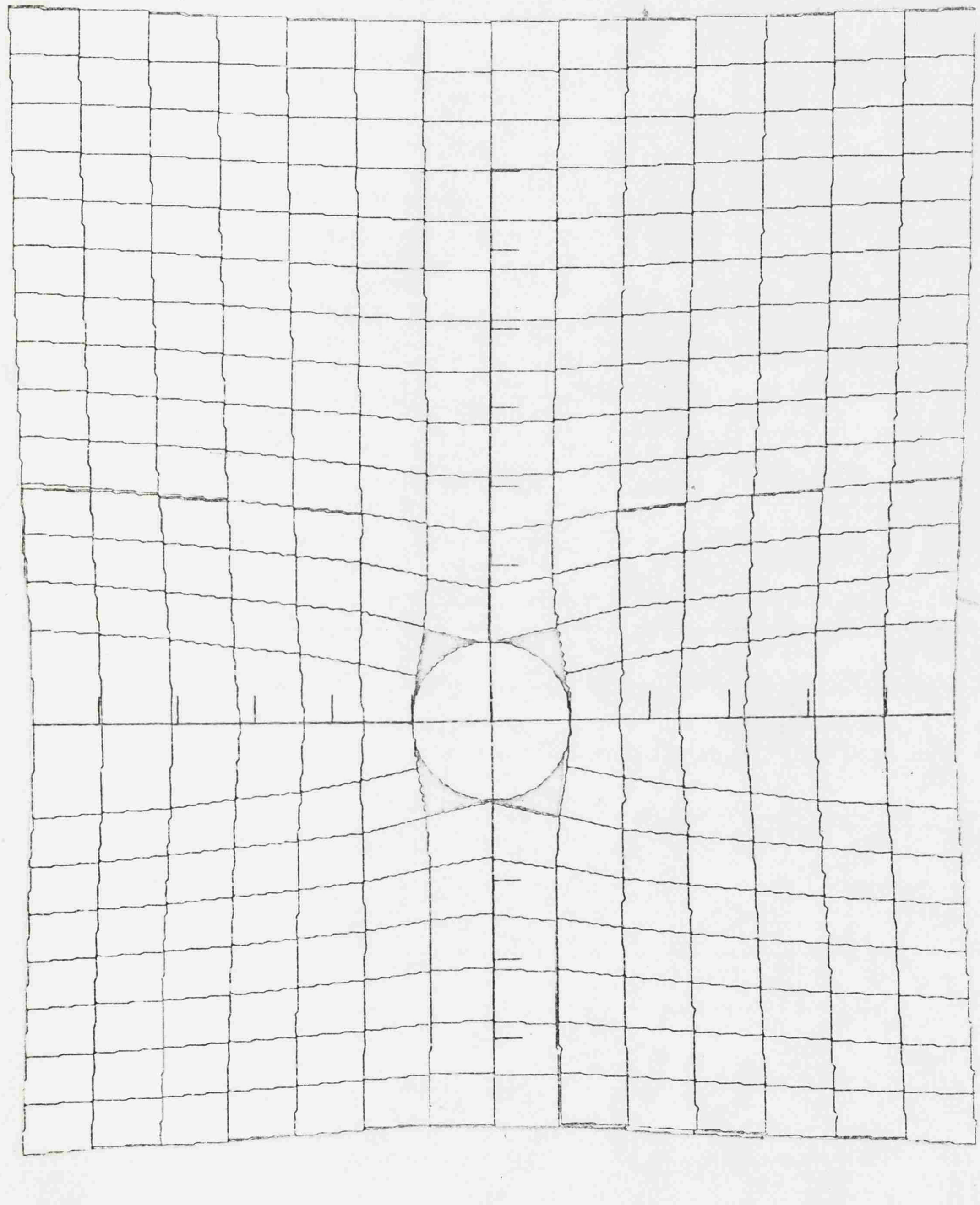


FIG 6.4.2(I)
T = 0.25

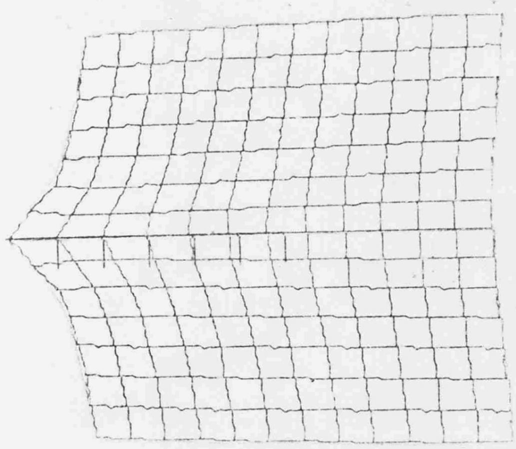


FIG 6.4.2(iii)
T = ·75

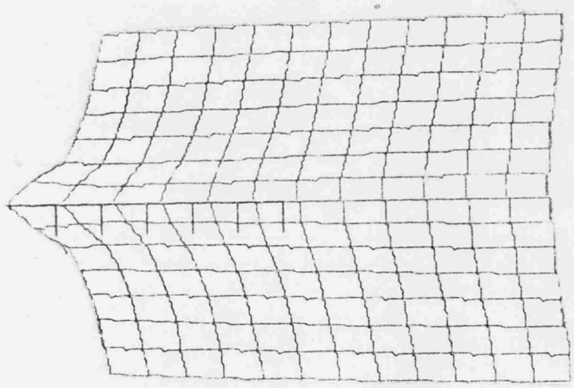


FIG 6.4.2(vi)
T = 1·0

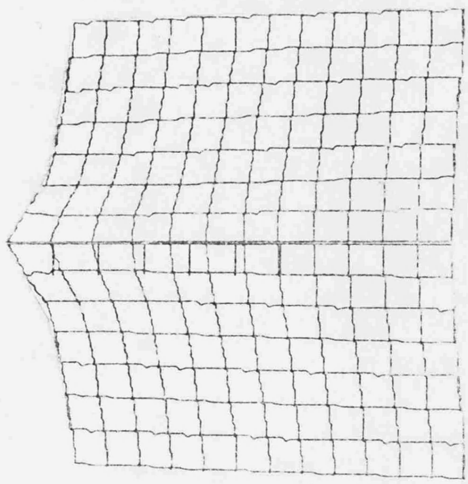


FIG 6.4.2(ii)
T = ·6

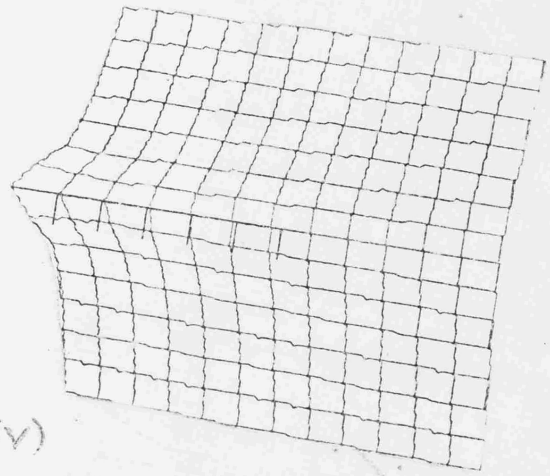


FIG 6.4.2(iv)
T = 0.8

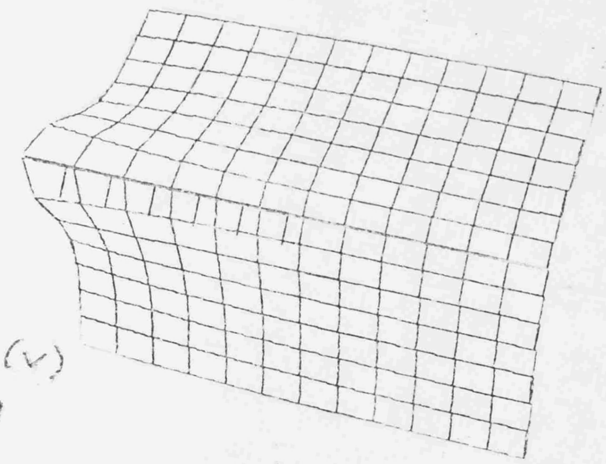


FIG 6.4.2(v)
T = 0.9

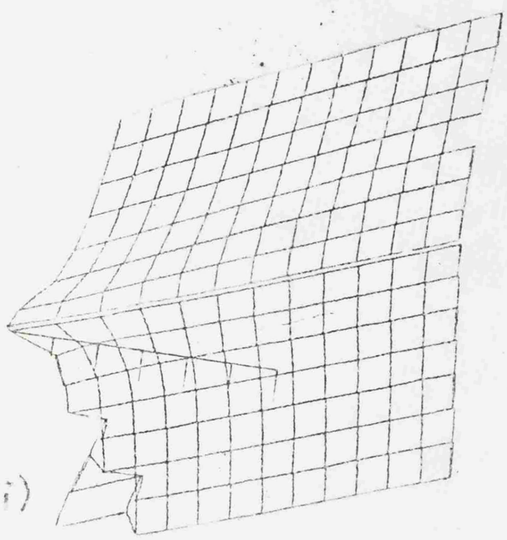


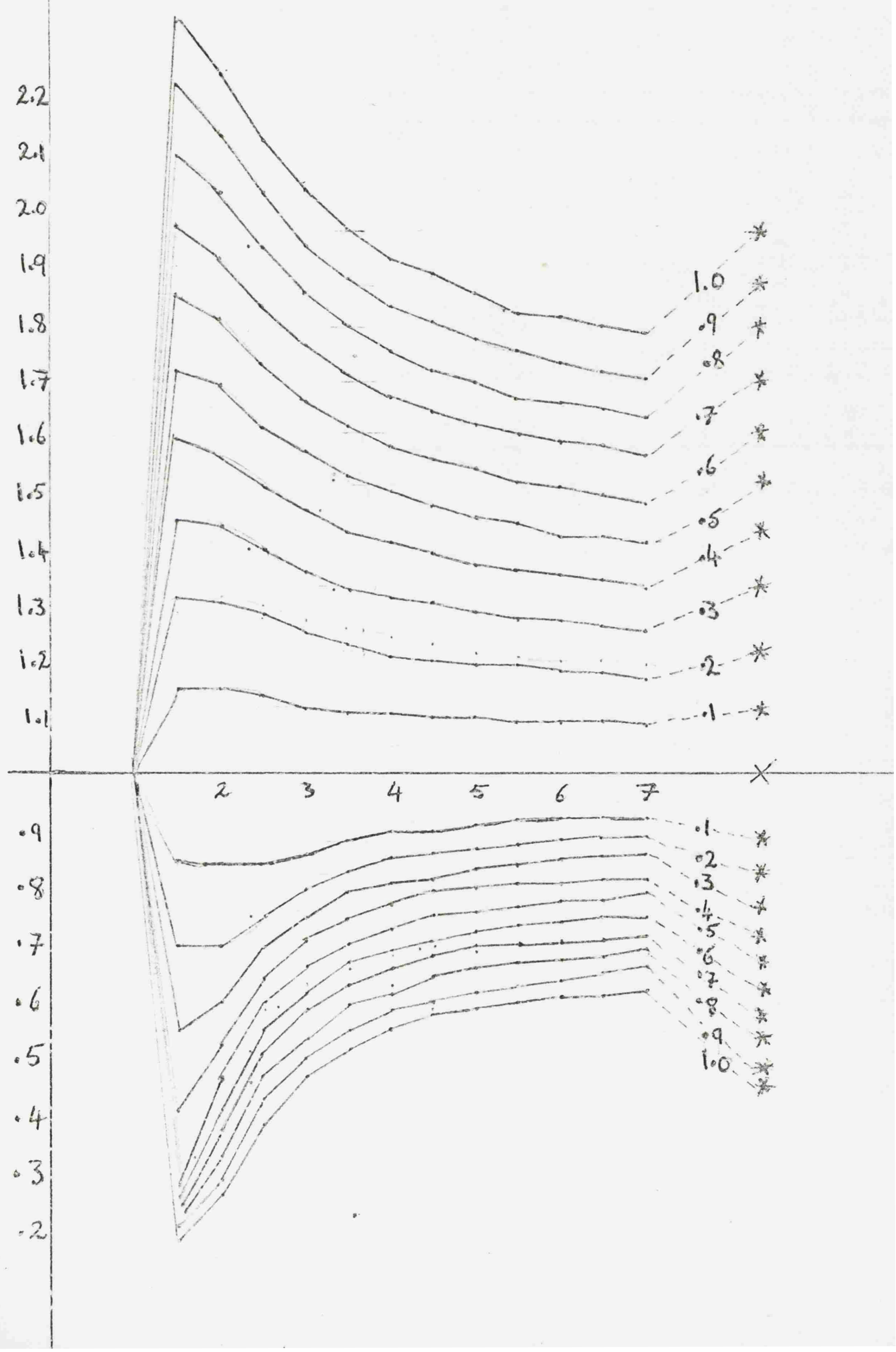
FIG 6.4.2(vii)
T = (1.0, 0.5)

Finally in this series of results which correspond to the problem discussed in Section 6.2, figure 6.4.3 is a graphical representation of the strain along the co-ordinate axes. The ordinate is the strain as measured by the change in length of the Ø.5 interstitial distance. The abscissas is the distance of the point in question from the origin.

The family of curves above the horizontal axis which are in tension, correspond to points on the physical X_1-axis. The curves are indexed by the tension applied at infinity. The position of the asterisk at the end of each curve indicates the value of the average strain as measured over the whole interval.

The family of curves below the horizontal axis correspond to points on the physical X_2-axis and are under compression.

FIG 6.4.3


2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
2
3
4
5
6
7
.9
.8
.7
.6
.5
.4
.3
.2
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

The results, as presented in figures 6.4.1, 6.4.2 and 6.4.3, are distinguished only in that they are consistent with reality. More precisely, they are consistent with how the real deformation fields are envisaged. That is to say the predicted deformation fields are qualitatively correct. The deformation fields agree in form with those corresponding to a classical small strain analysis.

The deformation fields presented in figure 6.4.4 also appear to be qualitatively correct. However, these results are obtained not from the asymptotics of Sections 5.5 or 6.2, but as a consequence of an assumption implicit in the programs. Figures 6.4.4 correspond to the case when a remotely applied shear is admitted. That is the T of equation 6.2.5 (or 5.5.36) is allowed to be complex. Indeed, the assumption implicit in the programs is simply that.

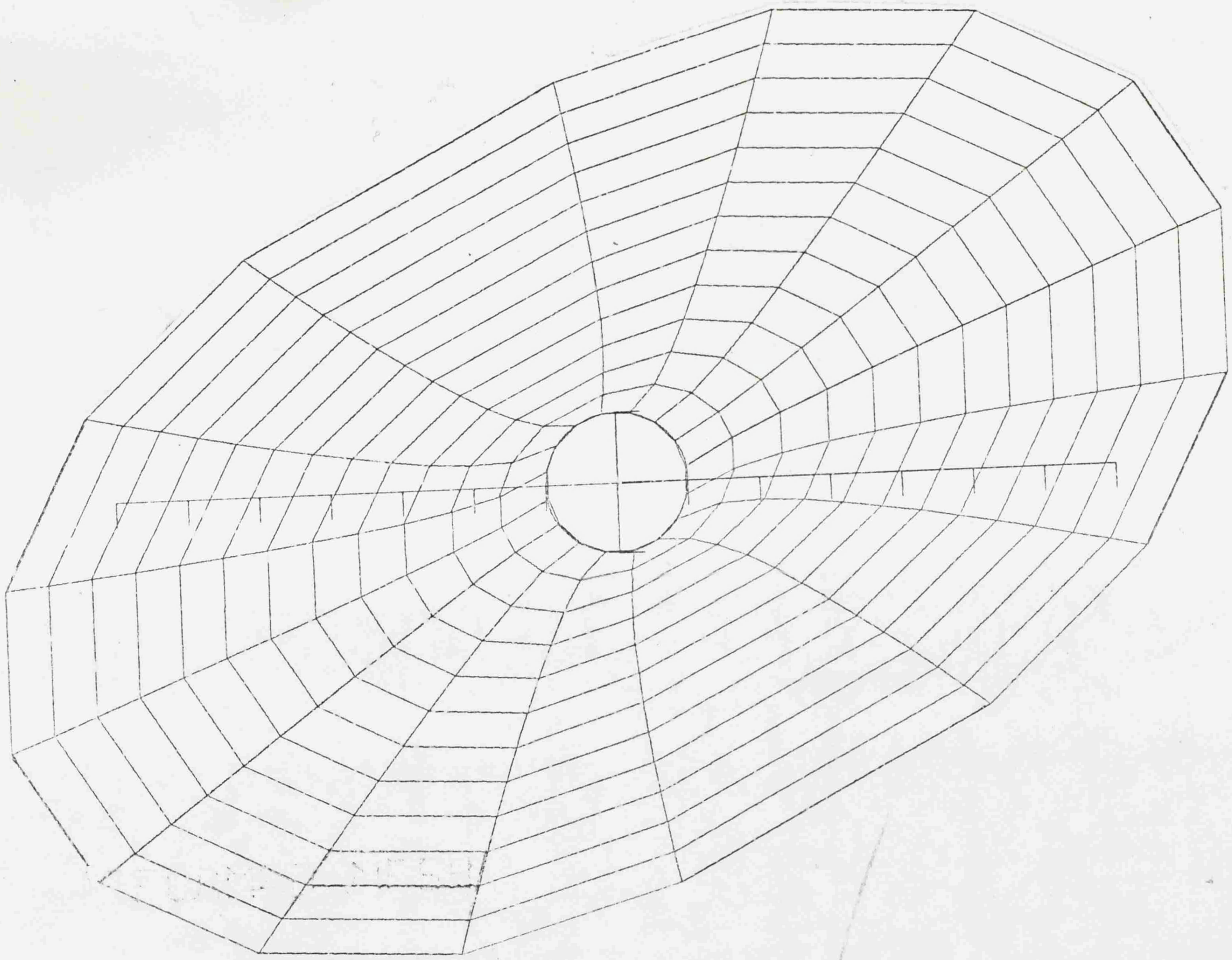


FIG 6.4.4(i) $T = (\cdot 25, \cdot 25)$

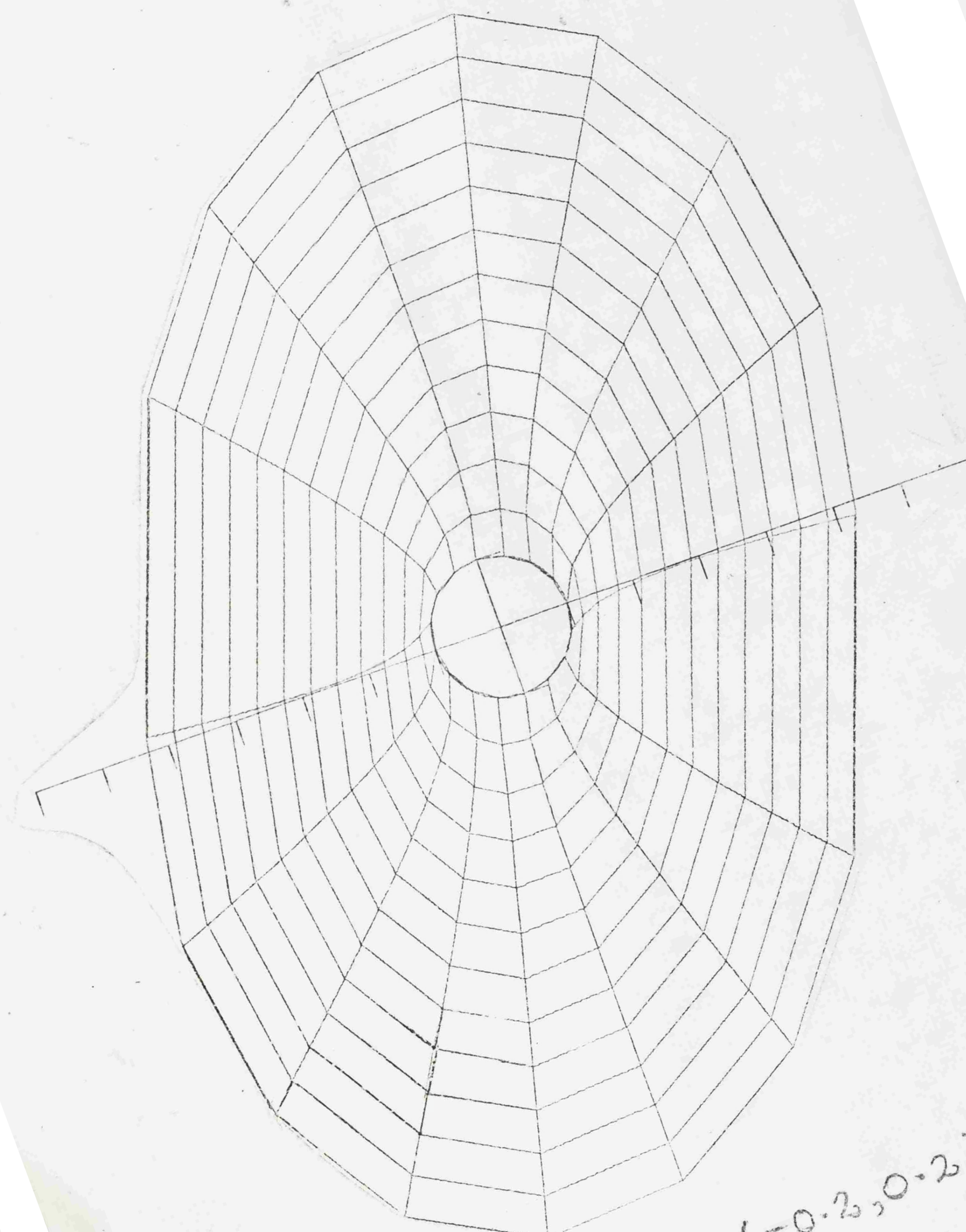


FIG 6.4.4(ii) T=(-0.2,0.2)

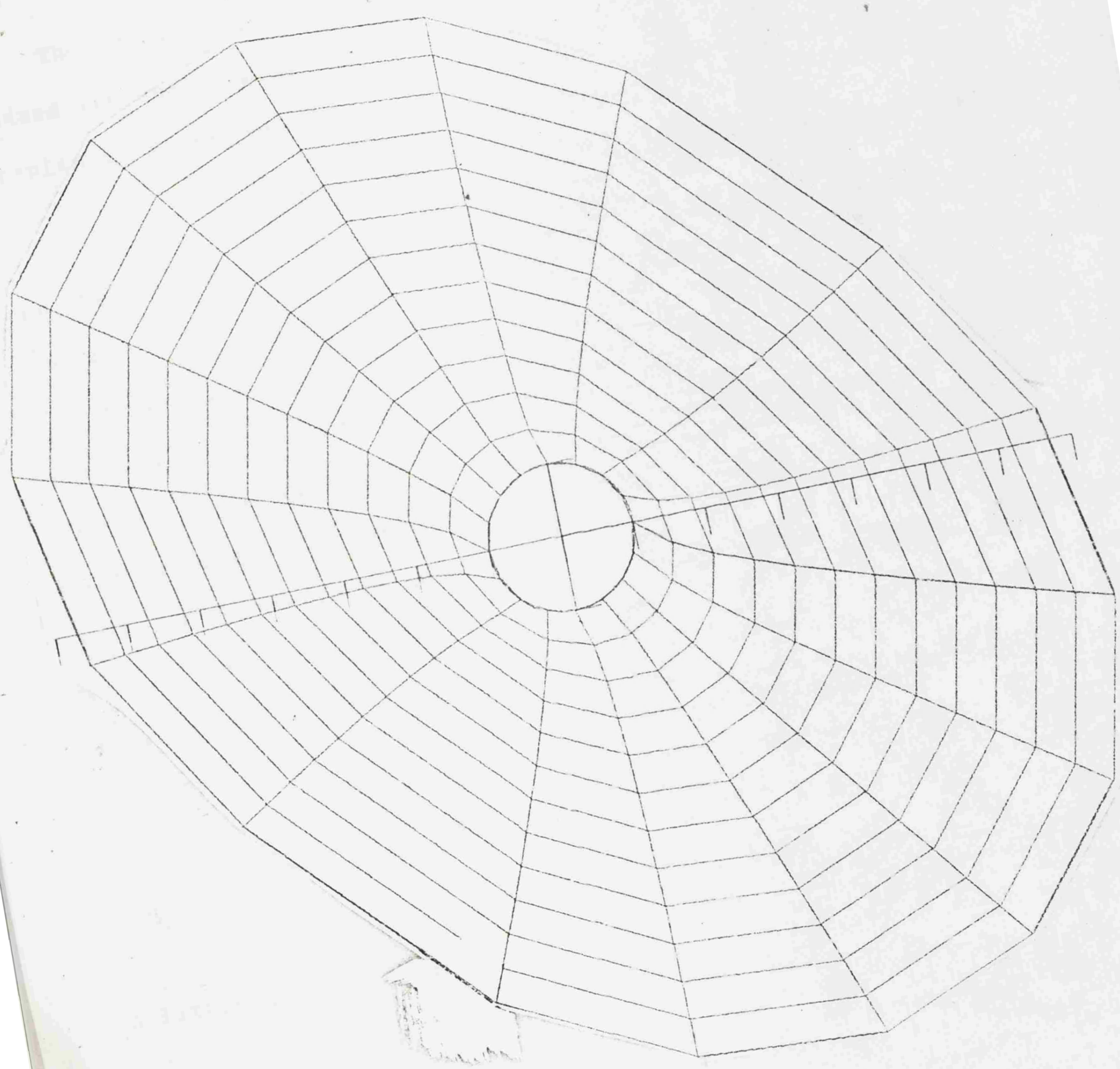


FIG 6.4.4(iii) T=(0.0,-0.2)

If the analysis resulting in 6.2.19, 6.2.33 and 6.2.30 is carried out for a complex T and an applied shear admitted, then the form for α(6.2.33) is unchanged, whereas the analysis yields a value for a_o of

$$a_o = \frac{1}{2}\frac{T}{/T/} + \frac{T}{8\mu}\frac{(\lambda+2\mu)}{(\lambda+\mu)} \quad . \qquad 6.4.1$$

The analysis leading to 5.5.11 may also be repeated for a complex T. Indeed the analysis is implied in that section. The result of that analysis results in an expression for α of that section, of

$$\frac{T}{/T/} + \frac{T(\lambda+2\mu)}{4\mu(\lambda+\mu)} \quad , \qquad 6.4.2$$

and the form for a_2 is unchanged. Inspection of the relationships between the respective coefficients indicates that 6.4.1 and 6.4.2 are equivalent.

However, there is some reason to doubt the results 6.4.1 and 6.4.2. Take 6.4.1 and the solution of Section 6.2 and consider the solution as represented by equations 6.2.37 with 6.2.38. Two cases are considered; firstly, when an applied tension is relaxed to zero and, secondly, when an applied pressure is relaxed to zero. In each case α of 6.2.34(vii) becomes zero in the limit, but a_o takes the value $\pm\frac{1}{2}$ depending on whether a tension or pressure is relaxed. The corresponding relaxed deformation fields are given by

$$x = \frac{a^2}{\bar{z}} \pm \frac{2\mu}{2(\lambda+2\mu)}\left[\bar{z} - \frac{a^2}{\bar{z}}\right] + \frac{\lambda+\mu}{\lambda+2\mu} I_3$$

with $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 6.4.3

$$I_3 = \pm\frac{1}{2}\left[\bar{z} - a^2/\bar{z}\right] \quad .$$

A little algebra results in the relaxed deformation fields

$$x = z \text{ when } a_o = +\frac{1}{2}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 6.4.4

$$x = \frac{a^2}{\bar{z}} - z \text{ when } a_o = -\frac{1}{2} \quad .$$

Thus the asymptotics with a complex T result in a form which predicts a deformation field resulting from a relaxed applied pressure, which is not the undeformed configuration. As such the results of the analysis would appear to be invalid at least in the limit as $/T/ \rightarrow 0$ when the term $T//T/$ ceases to be well defined.

The figures 6.4.4 which it has already been noted are qualitatively correct, are produced by adopting a form for a_o as in 6.2.34(vi) but allowing T to be complex.

The peculiar deformation field illustrated by figure 6.4.2(ii) will now be discussed. In that figure the deformation ceased to be single valued. Figures 6.4.5 illustrate further cases when the deformation field becomes multi-valued.

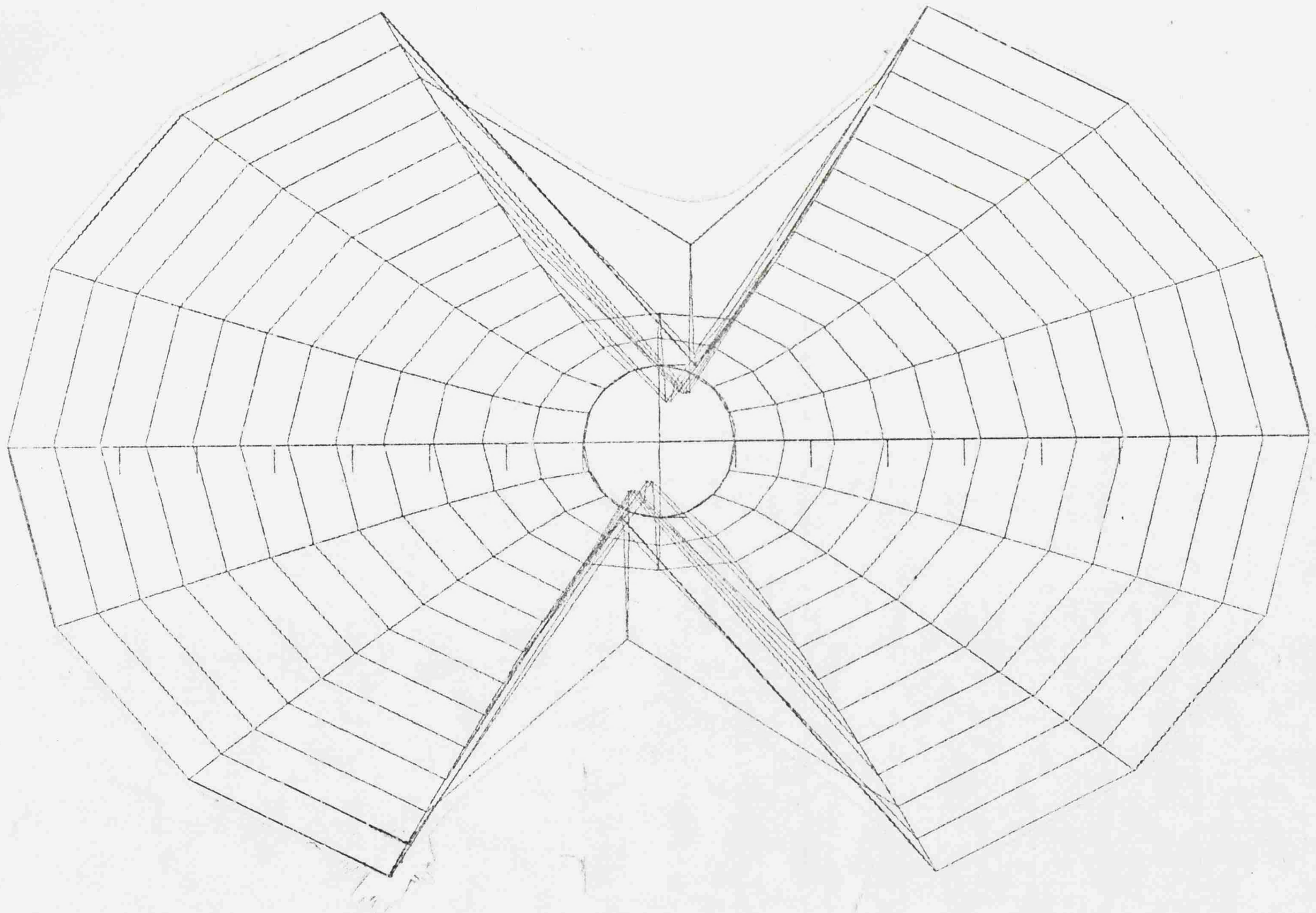


FIG 6.4.5(i) $T = 0.2$

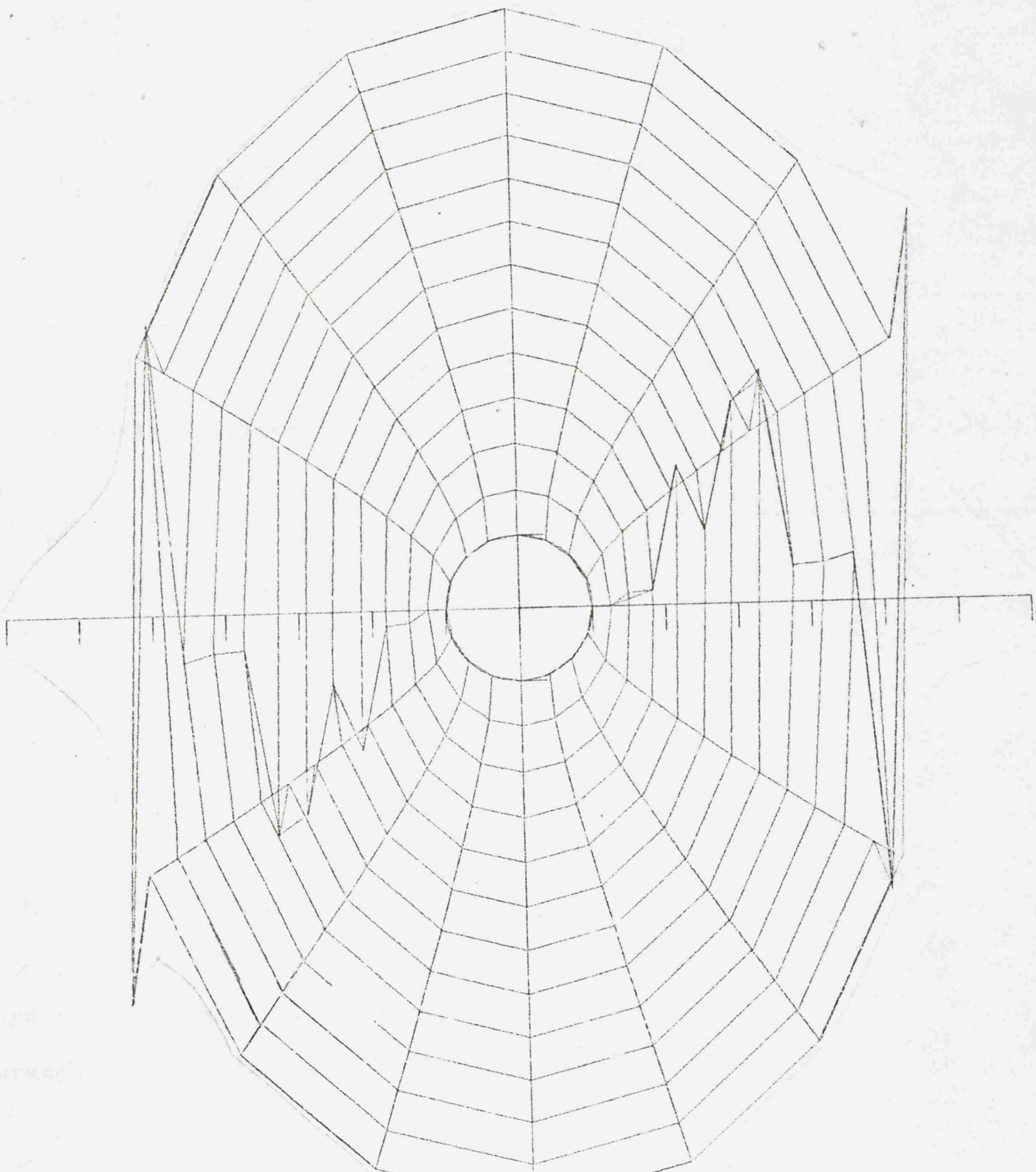


FIG 6.4.5(ii)   T = -0.2

In order to identify the cause of the obviously incorrect deformation fields of figures 6.4.5, the form 6.2.37 with 6.2.38 is used. These equations re-written with 'a' equal to 1 become

$$x = \frac{1}{\bar{z}} + \frac{2\mu}{(\lambda+2\mu)} \left[a_o z - \frac{a_o}{\bar{z}} - \frac{\alpha}{\bar{z}} + \alpha \bar{z} \right] + \frac{\lambda+\mu}{\lambda+2\mu} I_3 \qquad 6.4.5$$

with

$$I_3 = (a_o + \alpha\, \bar{z}^{-2})^{-\frac{1}{2}} \int_{\bar{z}^{-1}}^{z} (a_o + \alpha\, t^{-2})^{\frac{1}{2}}\, dt. \qquad 6.4.6$$

Of the terms involved in this expression for x only I_3 is capable of causing any difficulty. The expression

$$(a_o + \alpha\, t^{-2})^{\frac{1}{2}} \qquad 6.4.7$$

is obviously multi-valued and has a branch point where it assumes the value zero. When a_o and α are taken from 6.2.35(vi) and (vii) respectively with $\lambda = 1$ and $\mu = \frac{1}{2}$ as in the programs, the branch point has a locus of

$$t = \sqrt{\frac{-6T}{3+4T}}\,. \qquad 6.4.8$$

To proceed with the analysis, a real T is assumed. This corresponds to the situation as pictured in figures 6.4.5, and a detailed and particular analysis is possible. Figure 6.4.6 is a graphical representation of the location of the branch point, 6.4.8. The ordinate represents its modulus and a dotted line indicates that the branch point is real and a solid line that it is purely imaginary.

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APPENDIX A1 SMALL STRAIN ANALYSIS OF PLANE STRAIN RADIALLY SYMMETRIC PROBLEMS

In this section the notation as employed in many standard texts as "Advanced Mechanics of Materials" SEELY-SMITH, WILEY-TOPPAN is adopted.

Take

$$x_i \rightarrow x_i + u_i \ ,$$

with A1.1

$$u_i = x_i f(r)$$

as a specification of the deformation, where $r = (x_i x_i)^{\frac{1}{2}}$ is the polar radius. The constitutive relationship is taken in the form

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk} \ , \qquad \text{A1.2}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{A1.3}$$

defines the classical strain measure. $\underline{\sigma}$ is the conventional Cauchy stress as used in the main text.

The equilibrium equations in terms of the strain measure are

$$(\lambda+\mu)(\nabla \cdot \underline{u})_{,i} + \mu\nabla^2 u_i = 0 \ , \qquad \text{A1.4}$$

where ∇ is the conventional Laplacian operator. This equation may be written as

$$(\lambda+\mu)\ u_{j,ij} + \mu\ u_{i,jj} = 0 \ . \qquad \text{A1.5}$$

Now, given the deformation as in A1.1 the forms

$$u_{i,j} = f(r)\delta_{ij} + x_i x_j f'(r)/r$$

$$u_{j,j} = 2f(r) + rf'(r)$$

$$u_{i,jj} = 3x_i f'(r)/r + x_i f''(r)$$

and

$$u_{j,ij} = 2x_i f'(r)/r + x_i f'(r) + x_i f''(r)$$

are obtained. Substituting these into A1.5 yields

$$(\lambda+\mu)\ (3x_i\frac{f'(r)}{r} + x_i f''(r)) + \mu\ (3x_i\frac{f'(r)}{r} + x_i f''(r)) = 0 \ , \qquad \text{A1.6}$$

that is

$$x_i(\lambda+2\mu)\ (\frac{3f'(r)}{r} + f''(r)) = 0 \ . \qquad \text{A1.7}$$

Assuming that $\lambda + 2\mu \neq 0$ this may be solved to yield

$$f(r) = C + Dr^{-2} \ . \qquad \text{A1.8}$$